

On the Diophantine Equation $p^x + (p + 7)^y = z^2$

Suton Tadee*

*Department of Mathematics, Faculty of Science and Technology,
Thepsatri Rajabhat University, Thailand*

Received: 16 April 2026, Received in revised form: 17 August 2026, Accepted: 24 August 2026

Available online: 2 September 2026

Abstract

Background and Objectives: One interesting part of number theory is Diophantine analysis, which is the study of integer solutions to equations. Diophantine analysis investigates to answer whether a certain Diophantine equation is unsolvable, or solvable with the unique integer solution, finitely many integer solutions or infinitely many integer solutions. The Diophantine equations come in many forms. One form that has received attention is the exponential Diophantine equation of the form $a^x + b^y = z^2$, where a, b are fixed positive integers and x, y, z are non-negative integer variables. To find the non-negative integer solutions of the Diophantine equation, the Catalan's conjecture plays an important role. This Conjecture stated that $(a, b, x, z) = (3, 2, 2, 3)$ is the unique non-negative integer solution of the Diophantine equation $a^x - b^y = 1$, when a, b, x and z are non-negative integers such that $\min\{a, b, x, y\} > 1$, which remained unproven for over 150 years until it was proved by Mihăilescu in 2004. Over the past decade, the Diophantine equation of the form $p^x + (p + n)^y = z^2$, where n is a fixed positive integer, p is a prime and x, y, z are non-negative integers, has received considerable attention of many mathematicians in the study and search for solutions. In this paper, we solve the Diophantine equation for the case $n = 7$, which has never been studied before.

Methodology: The research and solutions presented here utilize mathematical proofs and fundamental concepts of number theory, such as prime numbers, congruences, and Legendre symbol. Here, some properties of prime number are applied to prove the theorems, such as: Let p be a prime number with $p \notin \{2, 7\}$. If the Diophantine equation $p^x + (p + 7)^y = z^2$ has a non-negative integer solution (x, y, z) and y is even, then $2(p + 7)^{y/2} = p^x - 1$. This will be useful to prove our main results. For the congruent method and Legendre symbol, we often use them in proofs to find contradictions.

Main Results: In this article, by using elementary methods, the Diophantine equation $p^x + (p + 7)^y = z^2$, where p is a prime number and x, y, z are non-negative integers, is investigated. The research results found that if $p = 2$, then the Diophantine equation has exactly two non-negative integer solutions. The solutions (x, y, z) are $(3, 0, 3)$ and $(4, 1, 5)$. Unfortunately, we are not yet able to find all non-negative integer solutions of the Diophantine equation for the case $p = 3$, but we can show the explicit non-negative integer solutions of the

Diophantine equation under the given conditions. If $p = 3$ and $y \equiv 0 \pmod{4}$, then the Diophantine equation has the unique non-negative integer solution. The solution (x, y, z) is $(1, 0, 2)$. For the case $p = 5$, we prove that the Diophantine equation has a unique non-negative integer solution, $(x, y, z) = (2, 2, 13)$. For the case $p = 7$, we prove that the Diophantine equation has the unique non-negative integer solution, which is $(x, y, z) = (4, 5, 735)$. For the case $p \equiv 5, 11, 13, 43, 45, 51 \pmod{56}$, if the Diophantine equation has a non-negative integer solution (x, y, z) , then there exists a non-negative integer t such that $y = 2t$ and $2(p + 7)^t = p^x - 1$. Moreover, we present some conditions for non-existence of the non-negative integer solutions to the Diophantine equation, namely, (1) if $p \equiv 15 \pmod{28}$, then the Diophantine equation has no non-negative integer solution. (2) if $p \equiv 5, 11, 13, 43, 45, 51 \pmod{56}$ and there exists a positive integer $n > 1$ such that n is not power of 2 and $p \equiv 1 \pmod{n}$, then the Diophantine equation has also no non-negative integer solution.

Conclusions: By using elementary properties of prime numbers, congruences, and Legendre symbol, we present explicit non-negative integer solutions (x, y, z) of the Diophantine equation $p^x + (p + 7)^y = z^2$ for some prime number p . Note that $p + 7$ doesn't necessarily have to be a prime number. Moreover, we show some conditions of non-existence of non-negative integer solutions to the Diophantine equation. In order to prove our main results, Mihăilescu's theorem was also applied. This method can be applied to find non-negative integer solutions to similar Diophantine equations. Furthermore, it was found that there are some cases that remain unanswered, such as: case $p \equiv 1 \pmod{8}$, which is an interesting open problem for further study and research.

Keywords: Diophantine equation; non-negative integer solution; prime number; congruence; Legendre symbol

*Corresponding author. E-mail: suton.t@lawasri.tru.ac.th

Introduction

Diophantine equations of the form $p^x + (p + n)^y = z^2$, where n is a fixed positive integer, p is a prime number, and x, y, z are non-negative integers, have garnered considerable interest due to their elementary mathematical properties. For examples, Burshtein (2018) analyzed the Diophantine equation $p^x + (p + 4)^y = z^2$, where $p > 3$ and $p + 4$ are prime numbers and concluded that the Diophantine equation has no positive integer solution. In the same year, Gupta *et al.* (2018) proved that the Diophantine equation $p^x + (p + 6)^y = z^2$, where p and $p + 6$ are prime numbers with $p \equiv 1 \pmod{6}$, has no non-negative integer solution. Oliveira (2018) found that the Diophantine equation $p^x + (p + 8)^y = z^2$, where $p > 3$ and $p + 8$ are prime numbers, has no positive integer solution. In 2019, Kumar *et al.* (2019) proved that the Diophantine equation $p^x + (p + 12)^y = z^2$, where p and $p + 12$ are prime numbers with $p \equiv 1 \pmod{6}$, has no non-negative integer solution.

Subsequently, Dokchan & Pakapongpun (2021) established that the Diophantine equation $p^x + (p + 20)^y = z^2$, where p and $p + 20$ are prime numbers, has no positive integer solution.

In 2022, Tadee (2022a) proved that if x is even, then the Diophantine equation $p^x + (p + 10)^y = z^2$, where p and $p + 10$ are prime numbers, has no non-negative integer solution. If y is even, then the Diophantine equation has exactly two non-negative integer solutions $(p, x, y, z) \in \{(3, 1, 0, 2), (3, 3, 2, 14)\}$. Moreover, Tadee (2022b) founded some conditions for non-existence of the non-negative integer solutions of the Diophantine equation $p^x + (p + 14)^y = z^2$, where p and $p + 14$ are prime numbers. In the same year, Pandichelvi & Vanaja (2022) studied the Diophantine equation $p^x + (p + 2)^y = z^2$, where p is a prime number with $p \equiv 1 \pmod{4}$ and the sum of the exponents x and y is 1, 2 and 3.

In 2023, Mina & Bacani (2023) proved that the Diophantine equation $p^x + (p + 5)^y = z^2$, where p is a prime number, has the unique positive integer solution $(p, x, y, z) = (5, 3, 2, 15)$ whenever $x \not\equiv 1 \pmod{12}$. Orosram (2023) studied the Diophantine equation $(p + n)^x + p^y = z^2$, where p and $p + n$ are prime numbers and n is a positive integer with $n \equiv 0 \pmod{4}$. Vesarachasart *et al.* (2023) showed that the Diophantine equation $p^x + (p + 12)^y = z^2$, where $p \neq 11$ and $p + 12$ are prime numbers, has no solution in positive integers x, y and z . Furthermore, in 2024, Wannaphan & Tadee (2024) studied the Diophantine equation $p^x + (p + 1)^y = z^2$, where p is a prime number. Some research results showed that if $p > 3$ and y is even, then the equation has no non-negative integer solution. For $p = 5$, the equation has the unique non-negative integer solution, namely, $(x, y, z) = (4, 3, 29)$. Recently, Jaimano *et al.* (2025) studied the Diophantine equation $p^x + (p + 30)^y = z^2$, where p is a prime number, and found that if $p = 29$, then the equation has the unique non-negative integer solution $(x, y, z) = (2, 1, 30)$, where y is not a multiple of 29^2 .

Motivated by the papers mentioned above, we investigate the non-negative integer solutions (x, y, z) of the Diophantine equation $p^x + (p + 7)^y = z^2$, where p is a prime number with some conditions.

Methodology

In the beginning of this section, we recall the Definitions of the quadratic residue and the Legendre symbol and we give its properties.

Definition 1. Let p be an odd prime number and let a be an integer with $\gcd(a, p) = 1$. If the congruence $z^2 \equiv a \pmod{p}$ has an integer solution, then a is said to be a quadratic residue of p . Otherwise, a is called a quadratic nonresidue of p .

Definition 2. Let p be an odd prime number and let a be an integer such that $\gcd(a, p) = 1$. The Legendre symbol (a/p) is defined by

$$(a/p) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue of } p. \end{cases}$$

Theorem 1. (Burton, 2010) Let p be an odd prime number and let a, b be integers that are relatively prime to p .

Then, the Legendre symbol has the following properties:

- (a) If $a \equiv b \pmod{p}$, then $(a/p) = (b/p)$.
- (b) $(a^2/p) = 1$.
- (c) $(a/p) \equiv a^{(p-1)/2} \pmod{p}$.
- (d) $(ab/p) = (a/p)(b/p)$.
- (e) $(1/p) = 1$ and $(-1/p) = (-1)^{(p-1)/2}$.

Theorem 2. (Burton, 2010) If p is an odd prime number, then

$$(2/p) = \begin{cases} 1 & \text{if } p \equiv 1, 7 \pmod{8} \\ -1 & \text{if } p \equiv 3, 5 \pmod{8}. \end{cases}$$

Theorem 3. (Burton, 2010) If $p \neq 3$ is an odd prime number, then

$$(3/p) = \begin{cases} 1 & \text{if } p \equiv 1, 11 \pmod{12} \\ -1 & \text{if } p \equiv 5, 7 \pmod{12}. \end{cases}$$

Theorem 4. (Mina & Bacani, 2023) If $p \neq 5$ is an odd prime number, then

$$(5/p) = \begin{cases} 1 & \text{if } p \equiv 1, 9, 11, 19 \pmod{20} \\ -1 & \text{if } p \equiv 3, 7, 13, 17 \pmod{20}. \end{cases}$$

Theorem 5. (Mina & Bacani, 2022) If $p \neq 7$ is an odd prime number, then

$$(7/p) = \begin{cases} 1 & \text{if } p \equiv 1, 3, 9, 19, 25, 27 \pmod{28} \\ -1 & \text{if } p \equiv 5, 11, 13, 15, 17, 23 \pmod{28}. \end{cases}$$

Next, we present some theorems about the Diophantine equations, which we need throughout this paper.

Theorem 6. (Mihăilescu's theorem) (Mihăilescu, 2004) The Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$, has the unique integer solution $(a, b, x, y) = (3, 2, 2, 3)$.

Theorem 7. (Samaapat *et al.*, 2025) Let x, y, z be non-negative integers and let n be a positive integer with $n \equiv 2 \pmod{3}$. Then all solutions of the Diophantine equation $3^x + n^y = z^2$ are

$$(n, x, y, z) \in \{(2, 0, 3, 3), (n, 1, 0, 2), (2, 2, 4, 5), (11, 5, 4, 122)\} \cup \{(n, 0, 1, \sqrt{n+1}) : \sqrt{n+1} \in \mathbb{Z}\}.$$

Theorem 8. (Nilsrakoo *et al.*, 2026) The Diophantine equation $p^x + (2p)^y = z^2$ has non-negative integer solutions (x, y, z) with $x < y$ if and only if p is a Mersenne prime. Moreover, all non-negative integer solutions (p, x, y, z) with $x < y$ are

$$(p, x, y, z) \in \{(3, 2, 3, 15), (2^q - 1, q + 1, q + 2, (2^q - 1)^{(q+1)/2}(2^{q+1} - 1))\},$$

where $2^q - 1$ is a Mersenne prime with q is an odd prime number. In particular, for a fixed Mersenne prime p , the equation has the unique non-negative integer solution (x, y, z) with $x < y$.

Theorem 9. (Nilsrakoo *et al.*, 2026) The Diophantine equation $p^x + (2p)^y = z^2$ has non-negative integer solutions (x, y, z) with $x > y$ if and only if $p = 2$ or p is a Fermat prime. Moreover, all non-negative integer solutions (p, x, y, z) with $x > y$ are

$$(p, x, y, z) = (2, 2\alpha + 3, \alpha, 3 \cdot 2^\alpha),$$

where α is a positive integer and

$$(p, x, y, z) \in \left\{ (3, 6, 4, 45), \left(2^{2^n} + 1, 2^{n+1} - 1, 2^{n+1} - 2, (2^{2^n} + 1)^{2^{n-1}} (2^{2^n-1} + 1) \right) \right\},$$

where $2^{2^n} + 1$ is a Fermat prime with $n \in \mathbb{N}_0$. In particular, for a fixed Fermat prime $p > 3$, the equation has the unique non-negative integer solution (x, y, z) with $x > y$.

Theorem 10. (Leu & Li, 2003) The Diophantine equation $2x^2 + 1 = 3^n$ has exactly three positive rational integral solutions, namely, $(x, n) = (1, 1), (2, 2)$ and $(11, 5)$.

Additionally, we prove the following lemma, which we will use to prove the main theorems.

Lemma 11. Let p be a prime number with $p \notin \{2, 7\}$. If the Diophantine equation $p^x + (p + 7)^y = z^2$ has a non-negative integer solution (x, y, z) and y is even, then $2(p + 7)^{y/2} = p^x - 1$.

Proof. Let x, y and z be non-negative integers such that $p^x + (p + 7)^y = z^2$ and $y = 2t$ for some non-negative integer t . Then $(z - (p + 7)^t)(z + (p + 7)^t) = p^x$. Since p is a prime number, there exists a non-negative integer u such that $z - (p + 7)^t = p^u$ and $z + (p + 7)^t = p^{x-u}$. It implies that $2(p + 7)^t = p^u(p^{x-2u} - 1)$. Assume that $u > 0$. Then $p|2(p + 7)^t$. This implies that $p = 2$ or $p = 7$, which is impossible since $p \notin \{2, 7\}$. Thus, we get $u = 0$. The equation becomes $2(p + 7)^t = p^x - 1$ or equivalently, $2(p + 7)^{y/2} = p^x - 1$.

Results

In this section, we present our results.

Theorem 12. The Diophantine equation $2^x + 9^y = z^2$ has only two non-negative integer solutions, namely, $(x, y, z) \in \{(3, 0, 3), (4, 1, 5)\}$.

Proof. Let x, y and z be non-negative integers such that $2^x + 9^y = z^2$ or $2^x + 3^{2y} = z^2$. By Theorem 7, we get $(x, y, z) \in \{(3, 0, 3), (4, 1, 5)\}$.

For the case $p = 3$, we cannot find the complete solutions yet, but we can show all non-negative integer solutions under the following condition.

Theorem 13. If $y \equiv 0 \pmod{4}$, then the Diophantine equation $3^x + 10^y = z^2$ has the unique non-negative integer solution, namely, $(x, y, z) = (1, 0, 2)$.

Proof. Let x, y and z be non-negative integers such that $3^x + 10^y = z^2$ and $y = 4t$ for some non-negative integer t . Then $3^x + 10^{2(2t)} = z^2$. Applying Lemma 11, we obtain $2(10)^{2t} = 3^x - 1$ or $2(10^t)^2 + 1 = 3^x$. By Theorem 10, we get $x = 0$ or $t = 0$. Assume that $x = 0$. Then, we have $2(10^t)^2 = 0$, which is impossible. Thus $t = 0$. It implies that $3 = 3^x$ and so $x = 1$. Hence, the solution is $(x, y, z) = (1, 0, 2)$.

Theorem 14. Let p be a prime number with $p \equiv 5, 11, 13, 43, 45, 51 \pmod{56}$. If the Diophantine equation $p^x + (p + 7)^y = z^2$ has a non-negative integer solution (x, y, z) , then there exists a non-negative integer t such that $y = 2t$ and $2(p + 7)^t = p^x - 1$.

Proof. Let x, y and z be non-negative integers such that $p^x + (p + 7)^y = z^2$. Since $p \equiv 5, 11, 13, 43, 45, 51 \pmod{56}$, we get $p \equiv 3, 5 \pmod{8}$ and $p \equiv 5, 11, 13, 15, 17, 23 \pmod{28}$. Assume that $x = 0$. Therefore $1 + (p + 7)^y = z^2$ or $z^2 - (p + 7)^y = 1$. It easy to check that $z > 1$ and $y > 0$. By Theorem 6, the equation has no solution for $y > 1$. Then $y = 1$ and so $z^2 = p + 8$. It follows that $z^2 \equiv 8 \pmod{p}$. Then $(8/p) = 1$. By Theorem 1, $(2^2/p)(2/p) = 1$ and so $(2/p) = 1$. By Theorem 2, we have $p \equiv 1, 7 \pmod{8}$, a contradiction since $p \equiv 3, 5 \pmod{8}$. Thus, we obtain $x \geq 1$. It implies that $z^2 = p^x + (p + 7)^y \equiv 7^y \pmod{p}$. Then, we have $(7^y/p) = 1$. By Theorem 1, we get $(7/p)^y = 1$. Since $p \equiv 5, 11, 13, 15, 17, 23 \pmod{28}$ and Theorem 5, it implies that $(-1)^y = 1$. Then, y is even. There exists a non-negative integer t such that $y = 2t$. By Lemma 11, we get $2(p + 7)^t = p^x - 1$.

Theorem 15. The Diophantine equation $5^x + 12^y = z^2$ has the unique non-negative integer solution, namely, $(x, y, z) = (2, 2, 13)$.

Proof. Let x, y and z be non-negative integers such that $5^x + 12^y = z^2$. By Theorem 14, there exists a non-negative integer t such that $y = 2t$ and $2 \cdot 12^t = 5^x - 1$. It is easy to check that $t > 0$. Assume that $t \geq 2$. Then $2 \cdot 12^t \equiv 0 \pmod{9}$ and so $5^x - 1 \equiv 0 \pmod{9}$. It implies that $x = 6k$, for some non-negative integer k . Then $2 \cdot 12^t = (5^{3k} - 1)(5^{3k} + 1)$. Since $5^{3k} \equiv (-1)^k \pmod{7}$, we have $7 | (5^{3k} - 1)(5^{3k} + 1)$. Then, we obtain that $7 | 2 \cdot 12^t$, a contradiction. Therefore, we have $t = 1$. It follows that $5^x = 25$ and so $x = 2$. Hence, the solution is $(x, y, z) = (2, 2, 13)$.

Theorem 16. The Diophantine equation $7^x + 14^y = z^2$ has the unique non-negative integer solution, namely, $(x, y, z) = (4, 5, 735)$.

Proof. Let x, y and z be non-negative integers such that $7^x + 14^y = z^2$. Assume that $x = y$. Then, we have $7^x(1 + 2^x) = z^2$. Since $1 + 2^x \equiv 2, 3, 5 \pmod{7}$, we have $\gcd(7^x, 1 + 2^x) = 1$. There exist two non-negative integers u and v such that $7^x = u^2$ and $1 + 2^x = v^2$. Then $v^2 - 2^x = 1$. It easy to check that $v > 1$ and $x > 1$. By Theorem 6, we get $v = 3$ and $x = 3$. Then $7^3 = u^2$, a contradiction. Thus, $x \neq y$. If $x < y$,

then $(x, y, z) = (4, 5, 735)$ is the unique non-negative integer solution to the Diophantine equation by Theorem 8. If $x > y$, then the Diophantine equation has no non-negative integer solution by Theorem 9.

Theorem 17. Let $p \neq 7$ be a prime number. If there exists a positive integer $n > 1$ such that n is not power of 2 and $p \equiv 1 \pmod{n}$, then the Diophantine equation $p^x + (p + 7)^{2y} = z^2$ has no non-negative integer solution.

Proof. Let $n > 1$ be a positive integer such that n is not power of 2 and $p \equiv 1 \pmod{n}$. Assume that there exist non-negative integers x, y and z such that $p^x + (p + 7)^{2y} = z^2$. By Lemma 11, we get $2(p + 7)^y = p^x - 1$. Since $p \equiv 1 \pmod{n}$, it implies that $2(p + 7)^t \equiv 2(8)^t \equiv 2^{3t+1} \pmod{n}$ and $p^x - 1 \equiv 0 \pmod{n}$. Then, we have $2^{3t+1} \equiv 0 \pmod{n}$ and so $n = 2^k$ for some positive integer k , a contradiction. Hence, the Diophantine equation $p^x + (p + 7)^{2y} = z^2$ has no non-negative integer solution.

By Theorem 14 and 17, we have the following theorem.

Theorem 18. Let p be a prime number with $p \equiv 5, 11, 13, 43, 45, 51 \pmod{56}$. If there exists a positive integer $n > 1$ such that n is not power of 2 and $p \equiv 1 \pmod{n}$, then the Diophantine equation $p^x + (p + 7)^y = z^2$ has no non-negative integer solution.

Corollary 19. The Diophantine equation $11^x + 18^y = z^2$ has no non-negative integer solution.

Corollary 20. The Diophantine equation $13^x + 20^y = z^2$ has no non-negative integer solution.

Theorem 21. Let p be a prime number with $p \equiv 15 \pmod{28}$. Then the Diophantine equation $p^x + (p + 7)^y = z^2$ has no non-negative integer solution.

Proof. Assume that there exist non-negative integer solutions x, y and z such that $p^x + (p + 7)^y = z^2$. Since $p \equiv 15 \pmod{28}$, we have $p \equiv 3 \pmod{4}$ and $p \equiv 1 \pmod{7}$. Now, we consider the following cases:

Case 1. $x = 0$. It implies that $1 + (p + 7)^y = z^2$ or $z^2 - (p + 7)^y = 1$. Then, we have $z > 1$. Suppose that $y = 0$. It follows that $z^2 = 2$, which is impossible. By Theorem 6, we get $y = 1$. Therefore $z^2 = p + 8$. Since $p \equiv 3 \pmod{4}$, we get $z^2 \equiv 3 \pmod{4}$. This is impossible since $z^2 \equiv 0, 1 \pmod{4}$.

Case 2. $x \geq 1$. It implies that $p^x + (p + 7)^y \equiv 7^y \pmod{p}$. Then, we have $z^2 \equiv 7^y \pmod{p}$. Therefore $(7^y/p) = 1$. By Theorem 1, $(7^y/p) = (7/p)^y$ and so $(7/p)^y = 1$. Since $p \equiv 15 \pmod{28}$ and Theorem 5, it implies that y is even. Then, there exists a non-negative integer t such that $y = 2t$. Thus, the equation becomes $p^x + (p + 7)^{2t} = z^2$. This is impossible since $p \equiv 1 \pmod{7}$ and Theorem 17.

Corollary 22. The Diophantine equation $43^x + 50^y = z^2$ has no non-negative integer solution.

Corollary 23. The Diophantine equation $71^x + 78^y = z^2$ has no non-negative integer solution.

Corollary 24. The Diophantine equation $127^x + 134^y = z^2$ has no non-negative integer solution.

Corollary 25. The Diophantine equation $211^x + 218^y = z^2$ has no non-negative integer solution.

Discussion

In this research, we solve the Diophantine equation $p^x + (p + 7)^y = z^2$, where p is a prime number that has not been studied before, as presented in the introduction. We use only basic concepts of number theory, such as: prime number congruence, quadratic residue and Legendre symbol. By proving several theorems, we can find all non-negative integer solutions to the equations for some prime numbers and discover the conditions under which the equations are unsolvable. However, there are still some cases for which we cannot find solutions, for example: $p \equiv 1 \pmod{8}$, making further study and research interesting.

Conclusions

By using the congruent method and properties of the Legendre symbol, we investigate the non-negative integer solutions (x, y, z) of the Diophantine equation $p^x + (p + 7)^y = z^2$, where p is a prime number under some conditions. We prove that for case $p = 2$, the equation has only two non-negative integer solutions, namely, $(x, y, z) \in \{(3, 0, 3), (4, 1, 5)\}$. If $p = 3$ and $y \equiv 0 \pmod{4}$, then the equation has the unique non-negative integer solution, namely $(x, y, z) = (1, 0, 2)$. If $p = 5$, then the equation has the unique non-negative integer solution, namely $(x, y, z) = (2, 2, 13)$. If $p = 7$, then the equation has the unique non-negative integer solution, namely, $(x, y, z) = (4, 5, 735)$. Finally, we present some conditions of non-existence of solutions to the equation, i.e., (1) $p \equiv 15 \pmod{28}$ and (2) $p \equiv 5, 11, 13, 43, 45, 51 \pmod{56}$ and there exists a positive integer $n > 1$ such that n is not power of 2 and $p \equiv 1 \pmod{n}$.

Acknowledgments

The author would like to thank the referee for the valuable suggestions and useful comments. This work was supported by Research and Development Institute and Faculty of Science and Technology, Thepsatri Rajabhat University, Thailand.

References

- Burshtein, N. (2018). The Diophantine equation $p^x + (p + 4)^y = z^2$ when $p > 3$, $p + 4$ are primes is insolvable in positive integers x, y, z . *Annals of Pure and Applied Mathematics*, 16(2), 283-286.
- Burton, D.M. (2010). *Elementary number theory*. 7th ed. New York: McGraw-Hill.
- Dokchan, R., & Pakapongpun, A. (2021). On the Diophantine equation $p^x + (p + 20)^y = z^2$, where p and $p + 20$ are primes. *International Journal of Mathematics and Computer Science*, 16(1), 179-183.

Gupta, S., Kumar, S., & Kishan, H. (2018). On the non-linear Diophantine equation $p^x + (p + 6)^y = z^2$.

Annals of Pure and Applied Mathematics, 18(1), 125-128.

Jaimano, K., Duanthip, N., Phrommarat, N., & Butsan, T. (2025). On the Diophantine equations

$p^x + (p + 30)^y = z^2$. *Journal of Science and Technology CRRU*, 4(2), 21-26. (in Thai)

Kumar, S., Gupta, D., & Kishan, H. (2019). On the Solutions of exponential Diophantine equation

$p^x + (p + 12)^y = z^2$. *International Transactions in Mathematical Sciences and Computers*, 11(1), 1-19.

Leu, M., & Li, G. (2003). The Diophantine equation $2x^2 + 1 = 3^n$. *Proceedings of the American Mathematical Society*, 131(12), 3643-3645.

Mihăilescu, P. (2004). Primary cyclotomic units and a proof of Catalan's conjecture. *Journal für die Reine und Angewandte Mathematik*, 572, 167-195.

Mina, R.J.S., & Bacani, J.B. (2022). On a Diophantine equation of type $p^x + q^y = z^3$. *Konuralp Journal of Mathematics*, 10(1), 55-58.

Mina, R.J.S., & Bacani, J.B. (2023). On the Diophantine equation $p^x + (p + 5)^y = z^2$, where p is odd prime. *Thai Journal of Mathematics*, 21(1), 67-75.

Nilsrakoo, W., Wichana, T., & Nilsrakoo, A. (2026). On the Diophantine equation $p^x + (2p)^y = z^2$, where p is prime. *Journal of Mathematics and Computer Science*, 41, 571-580.

Oliveira, F.N. (2018). On the solvability of the Diophantine equation $p^x + (p + 8)^y = z^2$ when $p > 3$ and $p + 8$ are primes. *Annals of Pure and Applied Mathematics*, 18(1), 9-13.

Orosram, W. (2023). On the Diophantine equation $(p + n)^x + p^y = z^2$ where p and $p + n$ are prime numbers. *European Journal of Pure and Applied Mathematics*, 16(4), 2208-2212.

Pandichelvi, V., & Vanaja, R. (2022). Inspecting integer solutions for an exponential Diophantine equation

$p^x + (p + 2)^y = z^2$. *Advances and Applications in Mathematical Sciences*, 21(8), 4693-4701.

Samaapat, N., Pomsiri, C., & Tadee, S. (2025). On the Diophantine equation $3^x + n^y = z^2$,

where $n \equiv 2 \pmod{3}$. *Journal of Science and Technology CRRU*, 4(2), 1-6. (in Thai)

Tadee, S. (2022a). On the Diophantine equation $p^x + (p + 10)^y = z^2$ when p and $p + 10$ are primes. *Udon*

Thani Rajabhat University Journal of Sciences and Technology, 10(2), 155-162. (in Thai)

Tadee, S. (2022b). On the Diophantine equation $p^x + (p + 14)^y = z^2$ where $p, p + 14$ are primes. *Annals of*

Pure and Applied Mathematics, 26(2), 125-130.

Vesarachasart, S., Eiemrawd, C., Thiengtham, S., & Mungkun, A. (2023). On the Diophantine equation

$p^x + (p + 12)^y = z^2$ where p and $p + 12$ are prime numbers. *Thai Journal of Science and Technology*, 12(1), 514-521. (in Thai)

Wannaphan, C. & Tadee, S. (2024). On the Diophantine equation $p^x + (p + 1)^y = z^2$, where p is prime. *Journal*

of Science Ladkrabang, 33(1), 1-7. (in Thai)