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การคงสภาพของฟลักซ์ซอยด์ควอนตัม $hc/2e$ ในวงแหวนตัวนำยวดยิ่งชนิดคลื่นเอส

Fluxoid Quantum $hc/2e$ Protected in an S-Wave Superconducting Ring

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บทคัดย่อ

วัตถุประสงค์และที่มา : คำถามถึงความเป็นไปได้การก้าวข้ามของฟลักซ์ซอยด์จากคาบการกวัดแกว่ง ลิตเติล-ปาร์ค $hc/2e$ ไปสู่คาบการกวัดแกว่ง hc/e เมื่อรัศมีของวงแหวนตัวนำยวดยิ่งชนิดคลื่นเอสมีขนาดลดลงได้รับการศึกษาในรายละเอียด

วิธีดำเนินการวิจัย : ใช้วิธีการคำนวณด้วยฟังก์ชันกรีนของกอร์คอฟเพื่อหาสมการอุณหภูมิวิกฤตและทำการวิเคราะห์ปรากฏการณ์การก้าวข้ามฟลักซ์ซอยด์ด้วยวิธีการคำนวณเชิงตัวเลข

ผลการวิจัย : ในการศึกษาปรากฏการณ์การก้าวข้ามฟลักซ์ซอยด์ทั้งในกรณีที่รัศมีมีค่าที่มากและกรณีที่รัศมีมีค่าจำกัดพบว่าทั้งสองกรณีอุณหภูมิวิกฤตยังคงสั่นด้วยคาบ $hc/2e$

สรุปผลการวิจัย : ผลลัพธ์ของการวิจัยพบว่าขัดแย้งกับงานเชิงทฤษฎีของเว่ยและโกลด์บาร์ทในบริบทของวงแหวนตัวนำยวดยิ่งบริสุทธิ์ในหนึ่งมิติ การศึกษานี้ได้เติมเต็มความเข้าใจที่ถูกต้องของภาวะเชิงคาบของฟลักซ์และการควอนไทเซชันของฟลักซ์ซอยด์ในอุณหภูมิวิกฤตของวงแหวนตัวนำยวดยิ่งชนิดคลื่นเอสที่ถูกทะลวงด้วยฟลักซ์แม่เหล็ก

คำสำคัญ : ตัวนำยวดยิ่ง ; ปรากฏการณ์ลิตเติล-ปาร์ค ; ปรากฏการณ์อาฮาโรนอฟ-โบห์ม

Abstract

Background and Objectives : A question of the possibility of the crossover of flux periodicity from the usual Little-Parks value of a fluxoid quantum $hc/2e$ to a period hc/e when the radius of an s-wave superconducting ring is reduced, is examined in detail.

Methodology : The method of Gor'kov Green's function is employed to calculate the superconducting critical temperature equation. The numerical analysis is studied to explore the fluxoid crossover phenomenon.

Main Results : We study the phenomenon of fluxoid crossover both in the cases of large- and finite-radius limits and found a period of oscillation of the critical temperature is always given by $hc/2e$. The obtained results show the fluxoid is strictly quantized in units of $hc/2e$.

Conclusions : Our analysis provides the results contrast to the theoretical work of Wei-Goldbart in the context of a one-dimensional superconducting clean ring. The study enhances the proper understanding of flux periodicity and fluxoid quantization in the critical temperature of an s-wave superconducting ring threaded by magnetic flux.

Keywords : superconductor ; Little-Parks effect ; Aharonov-Bohm effect

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Introduction

An experiment performed by Little & Parks in 1962 (Little & Parks, 1962) demonstrates that the fluxoid is quantized rather than the magnetic flux according to London's hypothesis (London, 1950) which known as the Little-Parks effect i.e., the phenomenon of the critical temperature oscillations of a thin-walled hollow superconducting cylinder in an axial magnetic field. The transition of the critical temperature with flux threading the hollow cylinder displays the parabolic profile having the $hc/2e$ flux periodicity at which the critical temperature reaches the highest value at the fluxoid being quantized to a discrete set of a fluxoid quantum $hc/2e$. The fundamental constants consisting of the Planck constant h, the speed of light c, and the electron charge e are combined in a form of flux quantum which can be realized macroscopically in a superconducting state. It is well understood that the quantity $2e$ stem from the pairing of electrons which so called the Cooper pair. Above the boundary line of the critical temperature where a superconducting state turns into a normal state and the Cooper pairs disintegrate into the normal electrons. The interference of the electron wavefunctions along the paths of multiply connected domain such as a metallic ring threaded by a magnetic flux, leads to the phenomenon of a geometric phase which is known as the Aharonov-Bohm (AB) effect (Aharonov & Bohm, 1959). The demonstration of the AB effect is the hc/e periodicity of the persistent current in a clean metallic ring (Webb *et al*, 1985).

A fundamental question of the possibility of the crossover of flux periodicity has been studied extensively how the period hc/e emerges in the superconducting state. For a clean ring of an s-wave superconductor, the Bogoliubov-de Gennes equations (De Gennes, 1966) in one dimension are solved with using the plane wave basis by Zhu & Wang (1994) (Zhu & Wang, 1994) while Loder *et al*, 2008 used the tight-binding model. Their analysis showed the AB phase arises from the mesoscopic effects. For a clean d-wave superconducting ring (Barash, 2008) the Cooper pair has node in the energy spectrum, so flux is divided in sectors centering around the periods $hc/2e$ and hc/e. An analysis in the context of the temperature driven crossover was examined by Wei and Goldbart (2008) (Wei & Goldbart, 2008) in a ring of an s-wave superconductor on the basis of the Gor'kov equations. They argued that the hc/e periodicity of the critical temperature oscillations emerges as the radius of the ring is made smaller than the superconducting coherence length. However, the recent work of Nisaisue & Krunavakarn (2023) using the linearized Ginzburg-Landau approach, showed the contrast results to the Wei-Goldbart argument. Therefore, we intend to re-investigate the work of Wei and Goldbart with the aims to confirm that the fluxoid quantization and the flux periodicity are still protected i.e., there is no the possibility of the emergence of the single-electron flux quantum hc/e in the superconducting state.

Gor'kov equations in an external magnetic field

We consider a superconducting ring of radius R that is subjected to a constant magnetic field $\vec{B}$ parallel to its axis; see Figure 1.

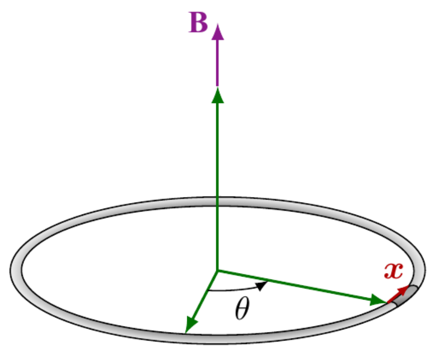

Figure 1 A superconducting ring of radius R, with a magnetic field threading the ring parallel to the ring axis

The magnetic field is related to a vector potential $\vec{A}$ by the relation $\vec{A}(r) = \frac{B}{2}(\hat{z} \times \vec{r})$, and the magnetic flux through the ring is given by $\Phi_B = \int \vec{B} \cdot d\vec{a}$. The Gor'kov equations (Fetter & Walecka, 1971) in one-dimensional ring model are

$$\left[i\hbar\omega_n + \frac{\hbar^2}{2M}\left(\partial_x - \frac{2i\pi\phi}{L}\right)^2 + \mu\right] g(x, x'; \omega_n) + \Delta(x) f^\dagger(x, x'; \omega_n) = \hbar\delta(x - x') \quad (1)$$

$$\left[-i\hbar\omega_n + \frac{\hbar^2}{2M}\left(\partial_x + \frac{2i\pi\phi}{L}\right)^2 + \mu\right] f^\dagger(x, x'; \omega_n) - \Delta^*(x) g(x, x'; \omega_n) = 0 \quad (2)$$

where M is mass of a charged particle, L is the circumference of the ring, $\phi = \frac{\Phi_B}{\Phi_0}$ is the dimensionless flux with $\Phi_0 = \frac{hc}{e}$ is the AB phase or the single-electron flux quantum, μ is the chemical potential, and the superconducting order parameter $\Delta(x)$ is defined self-consistently via

$$\varDelta^*(x) = \frac{V}{\beta}\sum_{\omega_n} f^\dagger(x, x; \omega_n) \tag{3}$$

with $\beta = 1/k_B T, T$ is temperature, $\omega_n = (2n+1)\pi k_B T$ is the Matsubara frequency for fermions, and V is BCS pairing strength. Using Fourier representation of g and $f^\dagger$ by expanding in the basis of the periodic function $\phi_n(x) \equiv e^{2i\pi n x/L}$ as follows

$$g(x, x'; \omega_n) = \frac{1}{L}\sum_{n_1,n_2} g_{n_1,n_2}(\omega_n)\, \phi_{n_1}(x)\phi_{n_2}(x') \tag{4}$$

$$f^\dagger(x, x'; \omega_n) = \frac{1}{L}\sum_{n_1,n_2} {f^\dagger}_{n_1,n_2}(\omega_n)\, \phi_{n_1}(x)\phi_{n_2}(x') \tag{5}$$

where n_1 and n_2 are integers labeling eigenstates. We consider the function $\varDelta(x) = \varDelta_0 e^{2\pi i m x/L}$, where $\varDelta_0$ is a constant and m is the vortex number. Assuming the temperature is near the critical temperature T_c, we can simplify our analysis by letting the gap function $\varDelta$ approaches zero. The equation for the critical temperature can then be expressed in the following form

$$1 = \frac{V}{\beta L}\sum_{\omega_n}\sum_{n_1} \frac{1}{(i\hbar\omega_n - \varOmega(n_1+m-\phi)^2+\mu)(-i\hbar\omega_n - \varOmega(n_1+\phi)^2+\mu)} \tag{6}$$

Employing the Poisson's summation formula $\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} dn f(n) e^{2i\pi k n}$, to turn the summation over n_1 to k, and we obtain the T_C equation as follows

$$1 = \frac{V}{\beta L \varOmega^2}\sum_{\omega_n}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} dx \frac{e^{2i\pi k(x-m/2)}}{((x-x_0)^2-\nu_n)((x+x_0)^2-\nu_n^*)} \tag{7}$$

where $x = (n_1 + m/2)$, $x_0 = \phi - m/2$, $\nu_n, \nu_n^* = (\mu \pm i\hbar\omega_n)/\Omega$, and $\Omega = \hbar^2/2MR^2$. The equation (7) has been used in the work of Wei-Goldbart to investigate the critical temperature oscillations in 1D rings.

Critical temperature formula

To calculate the self-consistent equation in the equation (7), we first note that the integrand of the equation (7) is a spin-singlet function which has the property of frequency symmetry. Then, we can simplify the equation and arrive at a more manageable form

$$1 = \frac{2V}{\beta L \Omega^2} \sum_{\omega_n \geq 0} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dx \frac{e^{2i\pi k(x-m/2)}}{((x-x_0)^2-\nu_n)((x+x_0)^2-\nu_n^*)} \tag{8}$$

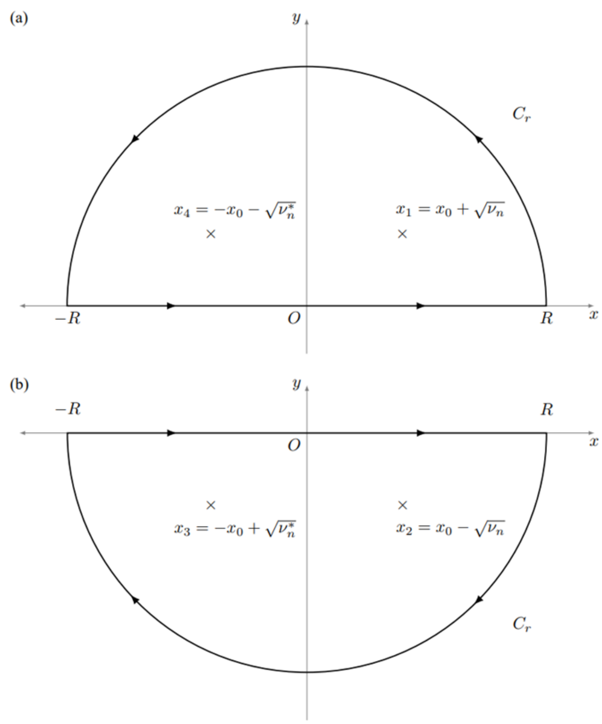

Figure 2 Four poles. (a) For $k \geq 0$, a closed contour C is composed of a straight line along the real axis and a semicircular arc C_r in the upper-half plane, enclosing two poles x_1 and x_4 in the upper-half complex plane. This contour is traversed in the counterclockwise direction. (b) For $k < 0$, a closed contour C is composed of a straight line along the real axis and a semicircular arc C_r in the lower-half plane, enclosing two poles x_2 and x_3 in the lower-half complex plane. This contour is traversed in the clockwise direction.

Performing the evaluation of the integral in the equation (8) by means of Cauchy's residue theorem. As shown in Figure 2., there are four poles in the complex plane. All of them are simple poles. For $k \geq 0$, we close the contour integral in the upper half plane and evaluate the residues at $x = x_0 + \sqrt{\nu_n}$ and $x = -x_0 - \sqrt{\nu_n^*}$.

For $k < 0$, we close contour integral in lower half plane and evaluate the residues at $x = x_0 - \sqrt{\nu_n}$ and $x = -x_0 + \sqrt{\nu_n^*}$. The final result is therefore

$$1 = \frac{\pi V}{\beta L \Omega^2} Re \sum_{\omega_n \geq 0} \left(\frac{1}{\sqrt{\nu_n}\left(\frac{\hbar\omega_n}{2\Omega} - ix_0^2 - ix_0\sqrt{\nu_n}\right)} + \sum_{k=1}^{\infty} \left(\frac{e^{2i\pi k\sqrt{\nu_n}}}{\sqrt{\nu_n}}\right) \Phi(\omega_n, \phi) \right) \tag{9.1}$$

where

$$\Phi(\omega_n, \phi) = \left(\frac{e^{2i\pi k\phi}}{\frac{\hbar\omega_n}{2\Omega} - ix_0^2 - ix_0\sqrt{\nu_n}} + \frac{e^{-2i\pi k\phi}}{\frac{\hbar\omega_n}{2\Omega} - ix_0^2 + ix_0\sqrt{\nu_n}} \right) \tag{9.2}$$

The equations (9.1) and (9.2) are evaluated properly in which we can see that the denominators of the equation (9.2) differed by the sign (plus and minus). We point out here that the formula derived by Wei-Goldbart is incorrect because they obtained only the term $\frac{\hbar\omega_n}{2\Omega} - ix_0^2 + ix_0\sqrt{\nu_n}$ which brings to the wrong analysis the behavior of the critical temperature oscillations.

Results

Large-radius limit

If the radius is large enough, we can neglect the summation over k in the equation (9.1) and arrive at the equation

$$1 = \frac{\pi V}{\beta L \Omega^2} Re \sum_{\omega_n \geq 0} \frac{1}{\sqrt{\nu_n}\left(\frac{\hbar\omega_n}{2\Omega} - ix_0^2 - ix_0\sqrt{\nu_n}\right)}$$

we simplify further by neglect the term $(x_0)^2$ with using the assumption that the Matsubara frequency ω_n is maximal at the Debye frequency ω_D and obeys the condition $\omega_D/\mu \ll 1$. Then the T_c equation can be obtained straightforwardly and is expressed in the form of the digamma function (Gradshteyn & Ryzhik, 2007)

$$ln(t) = \psi\left(\frac{1}{2}\right) - Re\left[\psi\left(\frac{1}{2} - \frac{ix_0\Gamma}{2\rho t}\right)\right] \tag{10}$$

where $t = T_c(\phi)/T_c(0)$ is the reduced critical temperature, $\Gamma \approx 1.76$ is the universal (BCS) constant and $\rho = R/\xi_0$ is the dimensionless radius with ξ_0 is the zero-temperature coherence length. Figure 3. shows the critical temperature oscillations in a superconducting ring with period $hc/2e$. The obtained result shows the periodic behavior of T_c as a function of fluxoid, which is known as the Little-Parks effect. This result is consistent with the 1962 experiment made by Little & Parks (1962) (Little & Parks, 1962).

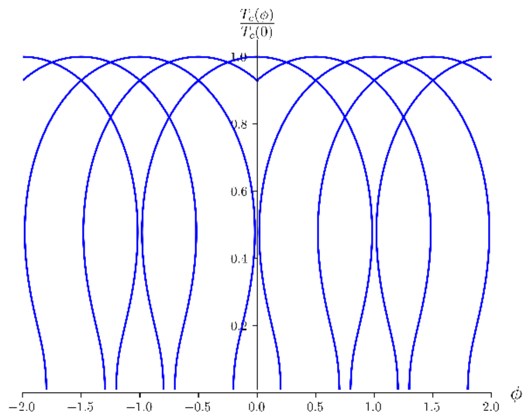

Figure 3 Critical temperature $t = T_c(\phi)/T_c(0)$ vs. flux ϕ for radius $R = 5\xi_0$ in Eq. (10). The oscillation period is found to be $hc/2e$.

Finite-radius modifications

The correction to T_c arises in the case of the finite radius where the summation over k in the equation (9) is important. As usual, we use the condition $\mu/\omega_D \gg 1$. The self-consistent equation can be taken in a form as follows

$$ln(t) \approx \psi\left(\frac{1}{2}\right) - Re\left[\psi\left(\frac{1}{2} - a\right) + \sum_{k=1}^{\infty} e^{2\pi ikN}\Psi(k, a, \phi)\right] \tag{11.1}$$

$$\Psi(k, a, \phi) = {}_2F_1\left(1, \frac{1}{2} - a, \frac{3}{2} - a; e^{-b}\right)\frac{e^{2\pi ik\phi}}{\frac{1}{2} - a}e^{-\frac{b}{2}} + {}_2F_1\left(1, \frac{1}{2} + a, \frac{3}{2} + a; e^{-b}\right)\frac{e^{-2\pi ik\phi}}{\frac{1}{2} + a}e^{-\frac{b}{2}}$$
$$-4e^{-\frac{c}{2}}{}_2F_1\left(1, \frac{1}{2}, \frac{3}{2}; e^{-c}\right) \tag{11.2}$$

where $t = T_c(\phi)/T_c(0)$, and the other parameters are $a = i\frac{x_0\Gamma}{2\rho t}$, $b = \frac{4\pi k\rho t}{\Gamma}$, $c = \frac{4\pi k\rho}{\Gamma}$. We have defined $N \equiv \sqrt{2MR^2\mu}$, and the function ${}_2F_1(\alpha, \beta, \gamma; z)$ is hypergeometric function. To compare our formula with the incorrect one of Wei-Goldbart, we plot t versus ϕ and show the results in Figure 4., where Figure (4a) is of Wei-

Goldbart and Figure (4b) is of our own. We can see that the oscillation patterns in Figure (4a) are divided into two classes depending on the vortex mode is either even or odd. For even number of vorticity the T_c is higher than that of the odd number. This led Wei-Goldbart to conclude that the flux periodicity is hc/e, namely, the AB flux period emerges in the small ring limit. However, Figure (4b) shows the behavior of the Little-Parks phenomenon with the flux periodicity is strictly $hc/2e$.

(a) (b)

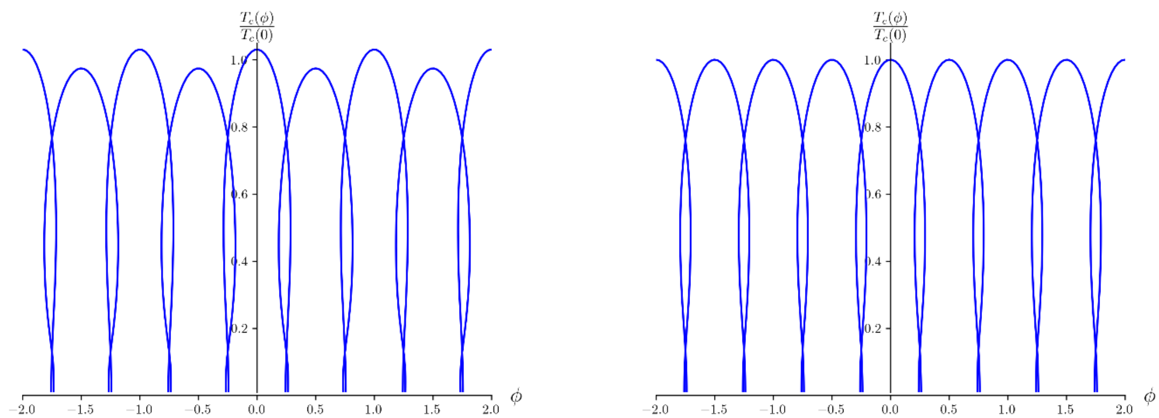

Figure 4 Critical temperature t vs. flux for $R = 1.5\xi_0$ in clean limit. The plots are made with $e^{2i\pi kN} = 1$ and with only the $k = 1$ term in Eq. (11.1) is retained. Figure (4a) reproduced the work of Wei-Goldbart. Meanwhile, Figure (4b) illustrates the result from numerical methods based on Eq. (11).

Discussion

In this study, the work of Wei-Goldbart has been examined in detail to explore the phenomena of the oscillations of the critical temperature T_c in a 1D superconducting ring threaded by an axial magnetic flux within the model of an s-wave pairing state, whether the single-electron flux quantum can be occurred in the superconducting state as the ring radius is comparable to the zero-temperature coherence length. The equation for the critical temperature in the large radius limit, equation (10), has the same form of Wei-Goldbart and there is no the emergence of the single-electron flux quantum presented in the Little-Parks oscillations.

For the finiteness of the radius, the correction to T_c is taken into account by the inclusion of the summation term as appeared in the equation (11.1). Surprisingly, the correction term differs from that of the Wei-Goldbart equation such that there is no the term like the cosine function which, in effect, leads to the incorrect analysis about the appearance of the single-electron flux quantum with a period hc/e.

Conclusions

The period of the critical temperature T_c oscillations in an s-wave superconducting ring is strictly given by $hc/2e$ regardless of the size of the ring radius. The flux periodicity also implies the fluxoid quantum $hc/2e$ is protected due to the parabolic shape of the oscillatory T_c reaches the highest value, for each vortex mode, at integral of $hc/2e$.

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