วารสารวิทยาศาสตร์บูรพา ปีที่ 29 (ฉบับที่ 3) กันยายน – ธันวาคม พ.ศ. 2567
BURAPHA SCIENCE JOURNAL Volume 29 (No.3) September – December 2024 บทความวิจัย

พีชคณิตของอะตอมมิคฟอร์มูล่าที่ก่อกำเนิดจากการส่งที่เซตไม่แปรเปลี่ยน

The Algebras of Atomic Formulas Generated by Mappings with an Invariant Set

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Received : 1 February 2024, Received in revised form : 24 September 2024, Accepted : 29 September 2024

Available online : 7 October 2024

บทคัดย่อ

วัตถุประสงค์และที่มา : อะตอมมิคฟอร์มูล่าซึ่งเป็นนิพจน์ทางคณิตศาสตร์ที่ถูกใช้ในทฤษฎีตรรกศาสตร์ร่วมสมัยนั้นถูกสร้างมาจากเทอมและสัญลักษณ์ความสัมพันธ์

วิธีดำเนินการวิจัย : อาศัยการส่งที่เซตไม่แปรเปลี่ยนบนเซตจำกัด $\{1, \ldots, n\}$ สำหรับจำนวนเต็มบวก n ใด ๆ อะตอมมิคฟอร์มูล่าที่ก่อกำเนิดจากการส่งดังกล่าวและตัวอย่างได้รับการนำเสนอ นอกจากนี้ยังพิสูจน์ว่าพีชคณิตของอะตอมมิคฟอร์มูล่าดังกล่าวสอดคล้องกับบางสัจพจน์โดยอาศัยการดำเนินการซุปเปอร์โพสิชัน R^n

ผลการวิจัย : โมนอยด์ของฟูลไฮเพอร์สับทิทิวชันสำหรับระบบเชิงพีชคณิตในบางชนิดที่นิยามบนของเซตของอะตอมมิคฟอร์มูล่าที่ก่อกำเนิดจากการส่งที่เซตไม่แปรเปลี่ยนได้รับการพิสูจน์ พีชคณิตสองโครงสร้างภายใต้การดำเนินการซุปเปอร์โพสิชัน R^n และการดำเนินการทวิภาค $\circ_r$ ซึ่งเป็นเครื่องมือสำคัญในทฤษฎีเอกลักษณ์ไฮเพอร์ได้ถูกสร้างขึ้น

สรุปผลการวิจัย : อะตอมมิคฟอร์มูล่าที่ก่อกำเนิดจากการส่งที่เซตไม่แปรเปลี่ยนได้รับการนำเสนอ พีชคณิตของฟอร์มูล่าสอดคล้องกับกฎการเปลี่ยนหมู่แบบซุปเปอร์

คำสำคัญ : ฟอร์มูล่า ; การส่งที่เซตไม่แปรเปลี่ยน ; การดำเนินการ ; โมนอยด์

Abstract

Background and Objectives : Atomic formulas, mathematical expressions used in a theory of classical logic, are combined from terms and relation symbols.

Methodology : Based on a mapping with an invariant set on a finite set $\{1, \ldots, n\}$ for a positive integer n. Atomic formulas generated by such mapping and some concrete examples are presented. By applying a superposition operation R^n, we show that the algebra of atomic formulas generated by a mapping with an invariant set satisfying some axioms is formed.

Main Results : The monoid of full hypersubstitutions for algebraic systems of some types defined on the set of atomic formulas generated by a mapping with an invariant set is proved. Two algebras of such formulas with respect to a superposition R^n and a binary operation $\circ_r$ which are tools to study a theory of hyperidentities are constructed.

Conclusions : Atomic formulas generated by mappings with an invariant set are given. Algebras of these formulas satisfty the superassociative law.

Keywords : formula ; a mapping with an invariant set ; operation ; monoid

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Introduction

It is well-known that formulas, which are algebraic expressions, always play a key role in logic and algebraic systems. Normally, by algebraic systems we mean a triple consisting of a nonempty set A, operations defined on A, and relations on A. A partially ordered semigroup is a basic example of algebraic systems. To explain algebraic properties of algebraic systems, one needs formulas. For example, on the set of real numbers $\mathbb{R}$, an expression $\exists x[x+1=3]$ is true because there is a real number 2 substituting in an equation $x+1=3$, which implies that this equation holds. In this matter, $\exists x[x+1=3]$ can be viewed as a formula in this aspect. By the definition, let τ be a type of operation symbols and τ' be a type of relation symbols. Recall from (Denecke, 2019) that a formula of type (τ,τ') is defined by the following steps:

1. An equation $s \approx t$ is a formula of type (τ,τ') if both s and t are terms of type τ.

2. If $t_1,\dots,t_{n_j}$ are terms of type τ and γ_j is an n_j-relation symbol, then $\gamma_j(t_1,\dots,t_{n_j})$ is a formula of type (τ,τ').

3. If F is a formula of type (τ,τ'), then $\neg F$ is a formula of type (τ,τ').

4. If F_1 and F_2 are formulas of type (τ,τ'), then $F_1 \vee F_2$ is a formula of type (τ,τ').

5. If F is a formula of type (τ,τ') and x_i is a variable form an alphabet X, then $\exists x_i[F]$ is a formula of type (τ,τ').

The symbol $\mathcal{F}_{(\tau,\tau')}(W_\tau(X))$ denotes the set of all formulas of type (τ,τ'). Actually, the formulas in the form 1. or 2. is said to be atomic. For more information and backgrounds about formulas and atomic formulas, see the papers (Joomwong & Phusanga, 2021; Kumduang & Leeratanavalee, 2021; Kumduang & Sriwongsa, 2023).

Let us consider some examples. On the set $\mathcal{F}_{(3,2)}(W_{(2)}(X))$ of formulas constructed from terms of type (2) with a ternary operation symbol f, a binary relation symbol γ and logical connectives, the following formulas belong to that set:

$$x_5 \approx f(x_5,x_5,x_5), \gamma\left(x_1, f(x_5,x_5,x_5)\right), \neg\left(f(x_2,x_1,x_2) \approx x_1\right), (x_1 \approx x_1) \vee \gamma\,(x_1,x_4).$$

However, the following formulas do not belong to that set: $x_1 \approx f(x_3,x_2), \gamma\,(x_1, f(x_5,x_5,x_5), x_2)$.

Kumduang and Leeratanavalee defined full formulas by applying the concept of full terms, terms constructed from a full transformation on a finite set $\overline{n} = \{1, \ldots, n\}$. See (Kumduang & Leeratanavalee, 2021). In fact, full terms and full formulas always play a central role in the theory of hyperidentities and solid varieties of algebras since these tools are ranges of mappings aiming to replace variables in identities by complicated steps.

We now recall the definition of a semigroup of transformations with an invariant set introduced in (Honyam, & Sanwong, 2011). For a fixed nonempty subset Y of a set X, the set $S(X,Y) = \{\alpha: X \to X \mid Y\alpha \subseteq Y\}$ whose elements are called transformations with an invariant set equipped with the usual composition of functions is a semigroup. For more details, we refer to (Sarkar & Singh 2022). Applying this structure, a particular class of full terms was given. In 2022, the concept of full terms with an invariant set was introduced in (Phuapong & Pookpienlert, 2022). Actually, an n-ary $S(\overline{n}, Y)$-full term of type τ_n is inductively defined in the following setting:

1. $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary $S(\overline{n}, Y)$-full term of type τ_n where $\alpha \in S(\overline{n}, Y)$,

2. If $t_1, \ldots, t_n$ are n-ary $S(\overline{n}, Y)$-full terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is an n-ary $S(\overline{n}, Y)$-full term of type τ_n.

3. The set $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ of all n-ary $S(\overline{n}, Y)$-full terms of type τ_n is the smallest set containing $f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)})$ and is closed under finite application of 2.

For example, let $\tau_3 = (3,3,3)$ be a type with three ternary operation symbols $\otimes, \boxtimes, \odot$. For a fixed subset $Y = \{1,3\}$ of $\overline{3}$, we have $\otimes\,(x_1,x_2,x_3), \boxtimes\,(x_1,x_3,x_1) \in W_{(3,3,3)}^{S(\overline{3},\{1,3\})}(X_3)$ because $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \in S(\overline{3}, \{1,3\})$. On the other hand, it is not difficult to see that $\otimes\,(x_2,x_1,x_1), \odot\,(x_2,x_2,x_2) \notin W_{(3,3,3)}^{S(\overline{3},\{1,3\})}(X_3)$ because $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$ do not belong to the set $S(\overline{3}, \{1,3\})$. Other classes of full terms can be found, for instance, in the paper (Kumduang, 2023A; Wattanatripop & Changphas, 2019).

The superposition operation S^n on the set $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ of all n-ary $S(\overline{n}, Y)$-full terms of type τ_n was mentioned in the paper (Phuapong & Pookpienlert, 2022; Wattanatripop & Changphas, 2021). In fact, it is a mapping

$$S^n : W_{\tau_n}^{S(\overline{n},Y)}(X_n)^{n+1} \to W_{\tau_n}^{S(\overline{n},Y)}(X_n)$$

defined by

1. $S^n\left(f_i\left(x_{\alpha(1)}, \dots, x_{\alpha(n)}\right), s_1, \dots, s_n\right) = f_i\left(s_{\alpha(1)}, \dots, s_{\alpha(n)}\right)$,
2. $S^n(f_i(t_1, \dots, t_n), s_1, \dots, s_n) = f_i\left(S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)\right)$.

It was mentioned in (Kumduang, 2023B) that the operation S^n defined on $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ satisfies the following identity:

$$S^n(S^n(t, s_1, \dots, s_n), u_1, \dots, u_n) = S^n\left(t, S^n(s_1, u_1, \dots, u_n), \dots, S^n(s_n, u_1, \dots, u_n)\right).$$

This equation also known as the superassociative law since it generalizes an associative law, meaning that if $n = 1$, the superassociative law is reduced to an associative law. The algebras satisfy the superassociative law have been investigated in many works, for example, see (Denecke & Hounnon, 2021; Dudek & Trokhimenko, 2021; Kumduang & Sriwongsa, 2023; Phuapong & Kumduang, 2021).

For continuation of the current work in this line, this paper aims to introduce a subclass of atomic formulas by using full terms with an invariant set and relation symbols. We also apply the superposition R^n given in (Kumduang & Leeratanavalee, 2021) to such atomic formulas. Furthermore, we also continue our investigation with describing full hypersubsitutions for algebraic systems of some types whose ranges are the union between the set of full terms with an invariant subset and the set of atomic formulas induced by full terms with an invariant set. The semigroup of these full hypersubstitutions under the binary associative operation is proved. Some concluding remarks and suggestions for future works are collected in the last section.

Methods

This section starts with the definition of atomic formulas generated by mappings with an invariant subset.

Definition 1 Let n and m be fixed positive integers. An atomic formula generated by full terms with an invariant set of type (τ_n, τ'_m) is defined in the following way:

1. An equation $s \approx t$ is an atomic formula generated by full terms with an invariant set of type (τ_n, τ'_m) if both s and t be full terms with an invariant set of type τ_n.

2. If $t_1, \dots, t_m$ are terms of type τ_n and γ_j is an m-ary relation symbol, then $\gamma_j(t_1, \dots, t_m)$ is an atomic formula generated by full terms with an invariant set of type (τ_n, τ'_m).

3.The set $\mathcal{F}_{(\tau_n,\tau'_m)}^{S(\overline{n},Y)}(X_n)$ of all atomic formulas generated by full terms with an invariant set of type (τ_n, τ'_m) is the smallest set which is closed under finite application of 2.

Some concrete examples of atomic formulas generated by full terms with an invariant set are now given.

Example 2 Let (τ_3, τ_2') be a type of algebraic systems where $\tau_3 = (3,3)$ with two ternary operation symbols f and g and $\tau_2' = (2)$ with a binary relation symbol ∇. If we now fix a subset $Y = \{1,3\} \subset \overline{3}$, then the following are some examples of atomic formulas generated by full terms with an invariant set Y of type (τ_3, τ_2') in the set $\mathcal{F}_{(\tau_3,\tau_2')}^{S(\overline{3},\{1,3\})}(X_3)$:

$$f(x_1,x_2,x_3) \approx g(x_3,x_3,x_3), g(x_1,x_1,x_1) \approx f(x_3,x_2,x_1), \nabla\big(g(x_3,x_3,x_3), f(x_1,x_1,x_1)\big)$$

because $\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 3 & 3 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} \in S(\overline{3},\{1,3\})$.

However, the following are not elements in the set $\mathcal{F}_{(\tau_3,\tau_2')}^{S(\overline{3},\{1,3\})}(X_3)$:

$$f(x_2,x_2,x_2) \approx f(x_3,x_3,x_3), g(x_1,x_3,x_2) \approx g(x_3,x_1,x_2), \nabla\big(g(x_3,x_2,x_1), f(x_2,x_3,x_1)\big)$$

because $\begin{pmatrix}1 & 2 & 3\\ 2 & 2 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} \notin S(\overline{3},\{1,3\})$.

Let β be a transformation with an invariant subset Y on a set $\overline{n}$. For each full term t with an invariant subset and atomic formula F with an invariant subset, we inductively define a full term t_β and an atomic formula F_β arising from β by the following steps.

1. If $t = f_i\big(x_{\alpha(1)}, \dots, x_{\alpha(n)}\big)$, then $t_\beta = f_i\big(x_{\beta(\alpha(1))}, \dots, x_{\beta(\alpha(n))}\big)$,
2. If $t = f_i(t_1, \dots, t_n)$, then $t_\beta = f_i\big((t_1)_\beta, \dots, (t_n)_\beta\big)$,
3. If $F = s \approx t$, then $F_\beta = s_\beta \approx t_\beta$,
4. If $F = \gamma_j(t_1, \dots, t_m)$, then $F_\beta = \gamma_j\big((t_1)_\beta, \dots, (t_m)_\beta\big)$.

Now we extend the definition of the superposition S^n defined on the set of full terms with an invariant subset to the set of all atomic formulas generated by full terms with an invariant set of type (τ_n, τ_m).

Definition 3 The superposition $R^n: \mathcal{F}_{(\tau_n,\tau_m')}^{S(\overline{n},Y)}(X_n) \times W_{\tau_n}^{S(\overline{n},Y)}(X_n)^{n+1} \to \mathcal{F}_{(\tau_n,\tau_m')}^{S(\overline{n},Y)}(X_n)$ where n is a positive integer is defined by the characteristic of atomic formulas generated by full terms with an invariant set as follows:

1. $R^n(s \approx t, s_1, \dots, s_n) = S^n(s, s_1, \dots, s_n) \approx S^n(t, s_1, \dots, s_n)$,
2. $R^n\big(\gamma_j(t_1, \dots, t_m), s_1, \dots, s_n\big) = \gamma_j\big(S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_m, s_1, \dots, s_n)\big)$.

To illustrate the process of computation of atomic formulas generated by full terms with an invariant set by the superposition R^n, let us consider the following example.

Example 4 Let (τ_3, τ_2) be a type of algebraic systems with one ternary operation symbol f and one relation symbol ∇. We also consider full terms $s_1 = f(x_1,x_2,x_3), s_2 = f(x_1,x_2,x_2), s_3 = f(x_2,x_3,x_2)$ in the set

$W_{(3)}^{S(\overline{3},\{2,3\})}(X_3)$ and atomic formulas $f(x_2,x_3,x_3)\approx f(x_1,x_2,x_2)$ and $\nabla(f(x_1,x_2,x_3),f(x_3,x_3,x_3))$ in the set $\mathcal{F}_{(\tau_3,\tau_2)}^{S(\overline{3},\{2,3\})}(X_3)$. Then we have that

1. $R^3(f(x_2,x_3,x_3)\approx f(x_1,x_2,x_2),s_1,s_2,s_3)=f(s_2,s_3,s_3)\approx f(s_1,s_2,s_2)$, which is equal to
   $f\big(f(x_1,x_2,x_2),f(x_2,x_3,x_2),f(x_2,x_3,x_2)\big)\approx f\big(f(x_1,x_2,x_3),f(x_1,x_2,x_2),f(x_1,x_2,x_2)\big)$.
2. $R^3(\nabla(f(x_1,x_2,x_3),f(x_3,x_3,x_3)),s_1,s_2,s_3)=\nabla(f(s_1,s_2,s_3),f(s_3,s_3,s_3))$, which is equal to
   $\nabla(f\big(f(x_1,x_2,x_3),f(x_1,x_2,x_2),f(x_2,x_3,x_2)\big),f\big(f(x_2,x_3,x_2),f(x_2,x_3,x_2),f(x_2,x_3,x_2)\big))$.

Results

We begin our results of the paper with construction of the algebra of atomic formulas generated by full terms with an invariant set.

From the previous section, the set $\mathcal{F}_{(\tau_n,\tau_m')}^{S(\overline{n},Y)}(X_n)$ of all atomic formulas generated by full terms with an invariant set of type (τ_n,τ_m') and the superposition R^n defined on this set are proposed. As a consequence, it is possible to construct the algebra

$$\mathcal{F}_{(\tau_n,\tau_m')}^{S(\overline{n},Y)}:=(\mathcal{F}_{(\tau_n,\tau_m')}^{S(\overline{n},Y)}(X_n),W_{\tau_n}^{S(\overline{n},Y)}(X_n),R^n,S^n)$$

consisting of two universe sets and two operations.

The following theorem shows that the operations R^n and S^n on the algebra $\mathcal{F}_{(\tau_n,\tau_m)}^{S(\overline{n},Y)}$ satisfies some important equations.

Theorem 5 The algebra $\mathcal{F}_{(\tau_n,\tau_m')}^{S(\overline{n},Y)}$ satisfies the following axiom:

$$R^n(R^n(F,s_1,\dots,s_n),t_1,\dots,t_n)=R^n(F,S^n(s_1,t_1,\dots,t_n),\dots,S^n(s_n,t_1,\dots,t_n))$$

for every $F\in\mathcal{F}_{(\tau_n,\tau_m')}^{S(\overline{n},Y)}(X_n), s_j,t_j\in W_{\tau_n}^{S(\overline{n},Y)}(X_n), j=1,\dots,n$.

Proof. We give a proof by the definition of an atomic formula F. In the first case, if F is an equation $s\approx t$ where s and t are full terms with an invariant set, then by the superassociativity of S^n, we have

$$\begin{aligned}&R^n(R^n(s\approx t,s_1,\dots,s_n),t_1,\dots,t_n)\\&=R^n(S^n(s,s_1,\dots,s_n)\approx S^n(t,s_1,\dots,s_n),t_1,\dots,t_n)\\&=S^n(S^n(s,s_1,\dots,s_n),t_1,\dots,t_n)\approx S^n(S^n(t,s_1,\dots,s_n),t_1,\dots,t_n)\\&=S^n(s,S^n(s_1,t_1,\dots,t_n),\dots,S^n(s_n,t_1,\dots,t_n))\approx S^n(t,S^n(s_1,t_1,\dots,t_n),\dots,S^n(s_n,t_1,\dots,t_n)\\&=R^n(s\approx t,S^n(s_1,t_1,\dots,t_n),\dots,S^n(s_n,t_1,\dots,t_n)).\end{aligned}$$

Assume that F is an expression $\gamma_j(u_1,\dots,u_m)$. Then we obtain

$$\begin{aligned}&R^n\big(R^n\big(\gamma_j(u_1,\dots,u_m),s_1,\dots,s_n\big),t_1,\dots,t_n\big)\\&=R^n\big(\gamma_j(S^n(u_1,s_1,\dots,s_n),\dots,S^n(u_m,s_1,\dots,s_n)),t_1,\dots,t_n\big)\end{aligned}$$

$= \gamma_j(S^n(S^n(u_1, s_1, \dots, s_n), t_1, \dots, t_n), \dots, S^n(S^n(u_m, s_1, \dots, s_n), t_1, \dots, t_n))$
$= \gamma_j(S^n(u_1, S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n)), \dots, S^n(u_m, S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n)))$
$= R^n(\gamma_j(u_1, \dots, u_m), S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n))$.

This finishes the proof.

Recall from (Kumduang & Leeratanavalee, 2021; Kunama & Leeratanavalee, 2023) that $Hyp^F(\tau_n, \tau'_m)$ denoted the set of all full hypersubstitutions for algebraic systems of type (τ_n, τ'_m), a mapping $\sigma: \{f_i | i \in I\} \cup \{\gamma_j | j \in J\} \to W^F_{\tau_n}(X_n) \cup \mathcal{F}^F_{(\tau_n, \tau'_m)}(X_n)$. Each full hypersubsitution σ can be extended to be a mapping $\hat{\sigma}$ which takes from $W^F_{\tau_n}(X_n) \cup \mathcal{F}^F_{(\tau_n, \tau'_m)}(X_n)$ to itself which can be defined by the following setting:

1. $\hat{\sigma}[f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})] = (\sigma(f_i))_\alpha$,
2. $\hat{\sigma}[f_i(t_1, \dots, t_n)] = S^n(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n])$,
3. $\hat{\sigma}[s \approx t] = \hat{\sigma}[s] \approx \hat{\sigma}[t]$,
4. $\hat{\sigma}[\gamma_j(t_1, \dots, t_m)] = R^m(\sigma(\gamma_j), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_m])$.

By this definition, the binary operation o_r on $Hyp^F(\tau_n, \tau'_m)$ is defined by $\sigma\ o_r\ \eta = \hat{\sigma} \circ \eta$ where $\circ$ denotes the usual composition of functions. It was proved that $(Hyp^F(\tau_n, \tau'_m), o_r, \sigma_{id})$ is a monoid where σ_{id} acts as an identity element with respect to o_r defined by $\sigma_{id}(f_i) = f_i(x_1, \dots, x_n)$ and $\sigma_{id}(\gamma_j) = \gamma_j(x_1, \dots, x_m)$.

Applying the concept of full hypersubstitutions, it is interesting to restrict the range of σ to the set of full terms with an invariant set and the set of atomic formulas with an invariant set, which we give the definition as follows:

Definition 6 A full hypersubstitution $\sigma \in Hyp^F(\tau_n, \tau'_m)$ for algebraic systems of type (τ_n, τ'_m) is said to be a full hypersubstitution with an invariant set if $\sigma(f_i) \in W^{S(\overline{n},Y)}_{\tau_n}(X_n)$ and $\sigma(\gamma_j) \in \mathcal{F}^{S(\overline{n},Y)}_{(\tau_n, \tau'_m)}(X_n)$. The set of all full hypersubstitutions with an invariant set of type (τ_n, τ'_m) is denoted by $Hyp^{S(\overline{n},Y)}(\tau_n, \tau'_m)$.

Our next aim is to show that the set $Hyp^{S(\overline{n},Y)}(\tau_n, \tau'_m)$ is a submonoid of $Hyp^F(\tau_n, \tau'_m)$ with respect to the binary operation o_r. For this, the following propositions are prepared.

Proposition 7 The extension of each full hypersubstitution with an invariant set of type (τ_n, τ'_m) maps from the set $W^{S(\overline{n},Y)}_{\tau_n}(X_n)$ to itself and maps from the set $\mathcal{F}^{S(\overline{n},Y)}_{(\tau_n, \tau'_m)}(X_m)$ to itself.

Proof Let t and F be a full term with an invariant subset and an atomic formula with an invariant set, respectively. We aim to show that $\hat{\sigma}[t] \in W^{S(\overline{n},Y)}_{\tau_n}(X_n)$ and $\hat{\sigma}[F] \in \mathcal{F}^{S(\overline{n},Y)}_{(\tau_n, \tau'_m)}(X_m)$. Now, we begin with $t = f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ where α belongs to $S(\overline{n}, Y)$. Thus, we have $\hat{\sigma}[f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})]$ is equal to $(\sigma(f_i))_\alpha$.

Since we know that for any mapping $\beta \in S(\overline{n}, Y)$ and any full term $t \in W_{\tau_n}^{S(\overline{n},Y)}(X_n)$, t_β contains in $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$. Our aim is obtained. If $t = f_i(t_1, \dots, t_n)$ and assume that $\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n] \in W_{\tau_n}^{S(\overline{n},Y)}(X_n)$, then by the paper (Kumduang, 2023B) we conclude $\hat{\sigma}[f_i(t_1, \dots, t_n)] = S^n(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) \in W_{\tau_n}^{S(\overline{n},Y)}(X_n)$. If F is an atomic formula $s \approx t$, then by an inductive step, we have $\hat{\sigma}[s \approx t]$ which equals $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is also an atomic formula with an invariant set. Finally, if F is an atomic formula $\gamma_j(t_1, \dots, t_m)$, and suppose that $\hat{\sigma}[t_j] \in W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ for $j = 1, \dots, m$, then by Theorem 5, we get $\hat{\sigma}[\gamma_j(t_1, \dots, t_m)] = R^m(\sigma(\gamma_j), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_m])$ belongs to $\mathcal{F}_{(\tau_n, \tau'_m)}^{S(\overline{n},Y)}(X_m)$.

Example 8 Let (τ_3, τ_2) be a type of algebraic systems with one ternary operation symbol f and one binary relation symbol ∇. Suppose that $\sigma \in Hyp^{S(\overline{3},\{2,3\})}(\tau_3, \tau_2)$ defined by $\sigma(f) = f(x_1, x_3, x_2)$ and $\sigma(\nabla) = f(x_1, x_3, x_3) \approx f(x_1, x_2, x_2)$. Then we have $\hat{\sigma}[f(x_2, x_3, x_3)] = (f(x_2, x_3, x_3))_{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}} = f(x_3, x_2, x_2) \in W_{\tau_3}^{S(\overline{3},\{2,3\})}(X_3)$. On the other hand, $\hat{\sigma}[f(x_2, x_3, x_3) \approx f(x_1, x_2, x_2)]$ is equal to the formula $(f(x_1, x_3, x_3))_{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}} \approx (f(x_1, x_2, x_2))_{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}}$ and thus $f(x_1, x_2, x_2) \approx f(x_1, x_3, x_3) \in \mathcal{F}_{(\tau_3, \tau_2)}^{S(\overline{3},\{2,3\})}(X_m)$.

Then we prove the following theorem.

Theorem 9 The set $Hyp^{S(\overline{n},Y)}(\tau_n, \tau'_m)$ is a subsemigroup of $Hyp^{F}(\tau_n, \tau'_m)$ under the binary operation o_r.

Proof Let σ and η be two full hypersubstitutions with an invariant set of type (τ_n, τ'_m). We show that under application the binary operation o_r from $Hyp^{F}(\tau_n, \tau'_m)$, $\sigma o_r \eta$ is a mapping in $Hyp^{S(\overline{n},Y)}(\tau_n, \tau'_m)$. To attain this, let f_i be an operation symbol of type τ_n and γ_j be a relation symbol of type τ'_m. From Proposition 7, it follows that $(\sigma o_r \eta)(f_i) = (\hat{\sigma} \circ \eta)(f_i) = \hat{\sigma}[\eta(f_i)] \in W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ and $(\sigma o_r \eta)(\gamma_j) = (\hat{\sigma} \circ \eta)(\gamma_j) = \hat{\sigma}[\eta(\gamma_j)] \in \mathcal{F}_{(\tau_n, \tau'_m)}^{S(\overline{n},Y)}(X_m)$.

Remark that in general, $Hyp^{S(\overline{n},Y)}(\tau_n, \tau'_m)$ is not a submonoid of $Hyp^{F}(\tau_n, \tau'_m)$ under the binary operation o_r because $\sigma_{id} \notin Hyp^{S(\overline{n},Y)}(\tau_n, \tau'_m)$. Namely, $\sigma_{id}(f_i) = f_i(x_1, \dots, x_n)$ does not necessary contains in $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ and $\sigma_{id}(\gamma_j) = \gamma_j(x_1, \dots, x_n)$ does not contains in $\mathcal{F}_{(\tau_n, \tau'_m)}^{S(\overline{n},Y)}(X_m)$.

Example 10 Let (τ_3, τ_3) be a type of algebraic systems with one ternary operation symbol f and one ternary relation symbol ∇. Let σ be a full hypersubstitution for algebraic systems which sends f to $f(x_1, x_3, x_2)$ and ∇ to $\nabla(f(x_2, x_3, x_3), f(x_2, x_2, x_3), f(x_1, x_2, x_3))$. Then $\sigma \in Hyp^{S(\overline{3},\{2,3\})}(\tau_3, \tau'_2)$.

Discussion

In this work, one of the subclasses of atomic formulas generated by full terms arising from a mapping with an invariant set on a finite set $\overline{n}$ is introduced. While examples and proofs of some propositions and theorems depend on a fixed subset Y of $\overline{n}$, which means that a characteristic of Y implies the expression of the set of atomic formulas. However, there are many symbols that appear in the work, we also paid an attention to use the standard notation as much as possible in the theory of hyperidentities and hypersubstitutions. For the definition of full hypersubstitutions with an invariant set for algebraic systems of type (τ_n, τ'_m), there is another approach to set a mapping σ. This mapping can be considered as having a domain that is a Cartesian product of operation symbols and relation symbols, taking elements in each component to the set of full terms with an invariant subset and the set of atomic formulas with an invariant set. See the paper (Denecke, 2019) for this particular method.

Conclusions

The set $\mathcal{F}^{S(\overline{n},Y)}_{(\tau_n,\tau'_m)}(X_n)$ of atomic formulas generated by full terms with an invariant set and operation symbols is proposed. Some concrete examples of these atomic formulas are also given. The superposition R^n defined on that set is superassociative and hence the algebra consisting of the set $\mathcal{F}^{S(\overline{n},Y)}_{(\tau_n,\tau'_m)\,)}(X_n)$ and $W^{S(\overline{n},Y)}_{\tau_n}(X_n)$ together with the operations R^n and S^n is constructed. It allows us to consider a subclass of full hypersubstitions whose range are universal sets of this algebra. The semigroup $Hyp^{S(\overline{n},Y)}(\tau_n, \tau'_m)$ with respect to the binary associative operation o_r is proved.

To continue the paper, the first direction is to apply the results to characterize of idempotency and regularity of elements in the semigroup $(Hyp^{S(\overline{n},Y)}(\tau_n, \tau'_m), o_r)$. With these characterizations, a process of computation of σ may be reduced in a natural way. Another way to develop the work is to study the power set of $\mathcal{F}^{S(\overline{n},Y)}_{(\tau_n,\tau'_m)}(X_n)$, apply a non-deterministic operation on such set, and try to examine the superassociativity.

Acknowledgment

This work was supported by Rajamangala University of Technology Rattanakosin, Thailand : Contract number C2/2567.

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