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ไฮเพอร์กรุปวัฏจักรบางชนิดกับสมบัติของดัชนีไฮเพอร์กรุปย่อย

Certain Cyclic Hypergroups with Property of Subhypergroup Indices

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บทคัดย่อ

วัตถุประสงค์และที่มา : กรุปวัฏจักรเป็นหนึ่งในกรุปที่มีความสำคัญที่มีการศึกษาในลักษณะเฉพาะต่าง ๆ สมบัติ(D) ของกรุป G คือสมบัติที่ว่าสำหรับกรุปย่อยที่แตกต่างกันของ G มีดัชนีที่แตกต่างกันใน G ด้วยการพิจารณาแยกกันทั้งในกรุปจำกัดและกรุปอนันต์ ได้ถูกพิสูจน์ว่ากรุป G เป็นกรุปวัฏจักรก็ต่อเมื่อกรุป G มีสมบัติ(D) นั่นหมายความว่ากรุป $(\mathbb{Z},+)$ และ $(\mathbb{Z}_n,+_n)$ มีสมบัติ(D) และไฮเพอร์กรุปเป็นลักษณะทั่วไปของกรุป ในงานวิจัยนี้จึงได้ขยายการศึกษาจากกรุปไปยังไฮเพอร์-กรุป เพื่อตรวจสอบความเป็นไฮเพอร์กรุปวัฏจักร และสมบัติ(D) ของไฮเพอร์กรุป $(\mathbb{Z},\circ_k)$ และ $(\mathbb{Z}_n,\circ'_k)$

วิธีดำเนินการวิจัย : ใช้การดำเนินการไฮเพอร์และผลแบ่งกั้นเซต $\mathbb{Z}$ และ $\mathbb{Z}_n$ เพื่อตรวจสอบความเป็นไฮเพอร์กรุปวัฏจักร การหาเอกลักษณ์ทั้งหมดและไฮเพอร์กรุปย่อยทั้งหมด พร้อมกับการหาตัวผกผันได้ เพื่อพิจารณาดัชนีของแต่ละไฮเพอร์กรุปย่อย และพิสูจน์การมีสมบัติ(D) ของไฮเพอร์กรุปทั้งสอง

ผลการวิจัย : ไฮเพอร์กรุป $(\mathbb{Z},\circ_k)$ และ $(\mathbb{Z}_n,\circ'_k)$ เป็นไฮเพอร์กรุปวัฏจักร ยกเว้น $k=0$ ใน $(\mathbb{Z},\circ_k)$ และไฮเพอร์กรุปทั้งสองมีสมบัติ(D)

สรุปผลการวิจัย : ในไฮเพอร์กรุปทั่วไป เราได้ว่าสมบัติ(D) ไม่เป็นลักษณะเฉพาะของไฮเพอร์กรุปวัฏจักร ซึ่งต่างจากผลในกรุป อย่างไรก็ตามดัชนีของไฮเพอร์กรุปย่อยของ $(\mathbb{Z},\circ_k)$ และ $(\mathbb{Z}_n,\circ'_k)$ มีผลเหมือนกับดัชนีของกรุปย่อยของ $(\mathbb{Z},+)$ และ $(\mathbb{Z}_n,+_n)$ ตามลำดับ

คำสำคัญ : ไฮเพอร์กรุปวัฏจักร ; ไฮเพอร์กรุปย่อย ; ดัชนีของไฮเพอร์กรุปย่อย

Abstract

Background and Objectives : Cyclic groups are one of the important groups which have been studied in many terms of characterizations. Property(D) of a group G is a property that distinct subgroups of G have distinct indices in G. Considering finite and infinite groups separately, it is proved that an arbitrary group is cyclic if and only if it has property(D). That means both groups $(\mathbb{Z},+)$ and $(\mathbb{Z}_n,+_n)$ have property(D). Also, hypergroups are generalizations of groups. In this research, we extend the study from groups to hypergroups in order to determine being cyclic hypergroups and property(D) of the hypergroups $(\mathbb{Z}, \circ_k)$ and $(\mathbb{Z}_n, \circ'_k)$.

Methodology : Using hyperoperations and partitions of sets $\mathbb{Z}$ and $\mathbb{Z}_n$ to determine being cyclic hypergroups. Finding all identity elements and all subhypergroups together with invertibilities to consider the index of each subhypergroup and prove having property(D) of both hypergroups.

Main Results : The hypergroups $(\mathbb{Z}, \circ_k)$ and $(\mathbb{Z}_n, \circ'_k)$ are cyclic except $k = 0$ in $(\mathbb{Z}, \circ_k)$ and both hypergroups have property(D).

Conclusions : In arbitrary hypergroups, property(D) is not a characterization of cyclic hypergroups, which is different from the results in groups. However, the indices of subhypergroups of $(\mathbb{Z}, \circ_k)$ and $(\mathbb{Z}_n, \circ'_k)$ have the same results as the indices of subgroups of $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +_n)$, respectively.

Keywords : cyclic hypergroup ; subhypergroup ; index of a subhypergroup

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Introduction

Cyclic groups are one of the most basic and important groups in group theory. We say a group G is cyclic if $G = \langle g \rangle = \{ g^n \mid n \in \mathbb{Z} \}$ for some $g \in G$. The additive group $\mathbb{Z}$ of integers is the unique infinite cyclic group and the additive group $\mathbb{Z}_n$ of integers modulo n is the unique finite cyclic group on n elements where $n \in \mathbb{Z}^+$ and $\mathbb{Z}^+$ is the set of positive integers. In 1934, F. Marty introduced hypergroups as generalizations of groups and made use of their properties to solve some problems of groups (Marty, 1934). P. Corsini and T. Vougiouklis studied and proved many results and theorems (Corsini, 1993; Vougiouklis, 1994). Also, B. Davvaz analyzed hyperstructures. We recall here some basic notions of hypergroup theory (Davvaz, 2013). Let H be a non-empty set and $\circ : H \times H \to P^*(H)$ be a hyperoperation (or join operation on H), where $P^*(H)$ is the family of non-empty subsets of H. For any two subsets A and B of H and $x \in H$, we define $A \circ B = \bigcup_{\substack{a \in A \\ a \in B}} a \circ b$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$. The singleton $\{x\}$ is generally represented by a member a. If N is a normal subgroup of a group G and $\circ_N$ is a hyperoperation on G defined by $x \circ_N y = xyN$ for all $x, y \in G$ then $(G, \circ_N)$ is a hypergroup (Corsini, 1993). It is known that all subgroups of cyclic groups are normal. Observe that if $G = \mathbb{Z}$, then $(G, \circ_N) = (\mathbb{Z}, \circ_k)$ where $x \circ_k y = x + y + k\mathbb{Z}$ for all

$x, y \in \mathbb{Z}$ and if $G = \mathbb{Z}_n$, then $(G, \circ_N) = (\mathbb{Z}_n, \circ'_k)$ where $x \circ'_k y = x +_n y +_n \left\langle \frac{n}{k} \right\rangle$ for all $x, y \in \mathbb{Z}_n$. In 2016, G. Omen and V. Slattum studied a characterization of the cyclic groups by subgroup indices (Omen & Slattum, 2016). They say a group G has property(D) if distinct subgroups of G have distinct indices in G. They proved that an arbitrary group G is cyclic if and only if it has property(D). The conclusion is that $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +_n)$ have property(D). In this research, we say that a hypergroup H has property(D) if distinct subhypergroups of H have distinct indices in H and we proved that a hypergroup $(\mathbb{Z}, \circ_k)$ is cyclic if $k \neq 0$ and a hypergroup $(\mathbb{Z}_n, \circ'_k)$ is cyclic if k is a positive divisor of n and showed that both hypergroups have property(D).

Methods

In this section, we will introduce some important concepts of abstract algebra and some definitions and theorems (Corsini, 1993; Triphop *et al.,* 2007; Zhan *et al*., 2011; Davvaz, 2013) which will be used as tools and methods to prove our results in the next section.

Proposition 1. Let $m, n, k \in \mathbb{Z}^+$. Then

i) $m\mathbb{Z} + k\mathbb{Z} = \gcd(m, k)\mathbb{Z}$.

ii) $m\mathbb{Z}_n + k\mathbb{Z}_n = \gcd(m, k)\mathbb{Z}_n$.

Lemma 2. If H is a subsemigroup of $(\mathbb{Z}, +)$ such that $H \cap \mathbb{Z}^- \neq \varnothing$ and $H \cap \mathbb{Z}^+ \neq \varnothing$, then $H = l\mathbb{Z}$ for some $l \in \mathbb{Z} \setminus \{0\}$.

Definition 3. Let H be a non-empty set and $\circ : H \times H \to P^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a *semihypergroup* if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$.

Definition 4. A semihypergroup $(H, \circ)$ is called a *hypergroup* if a $a \circ H = H \circ a = H$ for all $a \in H$.

Definition 5. A hypergroup $(H, \circ)$ is called *cyclic* if

$$H = \bigcup_{i \in \mathbb{Z}^+} h^i = h^1 \cup h^2 \cup h^3 \cdots \cup h^i \cup \cdots \text{ for some } h \in H$$

where $h^1 = \{h\}$ and $h^i = \underbrace{h \circ h \circ \cdots \circ h}_{i \text{ terms}}$.

Definition 6. A non-empty subset K of a hypergroup $(H, \circ)$ is a *subhypergroup of* H if $a \circ K = K \circ a = K$ for all $a \in K$.

Example 7. If H is a non-empty set and for all $x, y \in H,$ we define $x \circ y = H,$ then $(H, \circ)$ is a hypergroup, called the total hypergroup.

Example 8. If N is a normal subgroup of a group G and $\circ_N$ is a hyperoperation on G defined by $x \circ_N y = xyN$ for all $x, y \in G,$ then $(G, \circ_N)$ is a hypergroup.

Example 9. If G is a group and for all $x, y \in G,$ $\langle x, y \rangle$ denotes the subgroup generated by x and $y,$ then we define $x \circ y = \langle x, y \rangle.$ We obtain that $(G, \circ)$ is a hypergroup.

Definition 10. Let $(H, \circ)$ be a hypergroup and K be a subhypergroup of $H.$ We say that K is *invertible on the left* (*on the right*) if for all $x, y \in H,$ $x \in K \circ y$ $(x \in y \circ K)$ implies $y \in K \circ x$ $(y \in x \circ K).$

Definition 11. Let $(H, \circ)$ be a hypergroup and $(K, \circ)$ be a subhypergroup of $H.$ We say that K is *invertible* if it is invertible on the left and on the right.

Definition 12. Let $(H, \circ)$ be a hypergroup and $e \in H.$ We say e is an *identity* of $(H, \circ)$ if and only if $x \in (x \circ e) \cap (e \circ x)$ for all $x \in H.$

Definition 13. Let $(H, \circ)$ be a hypergroup, K be a subhypergroup of H and $x \in H.$ We say that $x \circ K$ $(K \circ x)$ is the *left* (*right*) *generalized coset of* K *in* H.

Definition 14. Suppose that H is a hypergroup containing at least one identity element and K is an invertible subhypergroup of $H.$ The number of all left generalized cosets of K in H is denoted by $[H:K]_l$ and the number of all right generalized cosets of K in H is denoted by $[H:K]_r$. If $[H:K]_l = [H:K]_r = n,$ then we say n is the *index of* K *in* H and denoted by $[H:K].$

Results

In this section, we study the hypergroups $(\mathbb{Z}, \circ_k)$ and $(\mathbb{Z}_n, \circ'_k)$ and prove both hypergroups are cyclic except $k \neq 0$ in $(\mathbb{Z}, \circ_k).$ We also determine their property(D).

First, we focus on certain hypergroups extended from all cyclic groups and determine when those hypergroups are cyclic hypergroups. Recall from Example 8, if N is a normal subgroup of a group G and $\circ_N$ is a hyperoperation on G defined by $x \circ_N y = xyN$ for all $x, y \in G,$ then $(G, \circ_N)$ is a hypergroup.

Theorem 1. Let $k \in \mathbb{Z}^{+} \cup \{0\}$. Then the following statements are true.

i) If $k = 0,$ then $(\mathbb{Z}, \circ_k)$ is not cyclic.

ii) If $k \in \mathbb{Z}^{+},$ then $(\mathbb{Z}, \circ_k)$ is cyclic.

Proof. i) Let $k = 0.$ Then $x^n = \underbrace{x \circ_k x \circ_k \cdots \circ_k x}_{n\ terms} = nx + k\mathbb{Z} = \{nx\}$ where $n \in \mathbb{Z}^{+}.$ Thus

$$\bigcup_{n\in\mathbb{Z}^{+}} x^n = \bigcup_{n\in\mathbb{Z}^{+}} \{nx\} = \{nx \mid n \in \mathbb{Z}^{+}\}.$$

This implies that $\bigcup_{n\in\mathbb{Z}^{+}} x^n \neq \mathbb{Z}$ for all $x \in \mathbb{Z}.$ Hence $(\mathbb{Z}, \circ_k)$ is not cyclic.

ii) Let $k \in \mathbb{Z}^{+}.$ Then there exists $1 \in \mathbb{Z}$ such that $1^n = n + k\mathbb{Z}$ where $n \in \mathbb{Z}^{+}$ such that $n \geq 2.$ Thus

$$\bigcup_{i=1}^{k+1} 1^i = \{1\} \cup \bigcup_{i=2}^{k+1} 1^i = \{1\} \cup \bigcup_{i=0}^{k-1} (i + k\mathbb{Z}) = \{1\} \cup \mathbb{Z} = \mathbb{Z}.$$

This implies that $(\mathbb{Z}, \circ_k)$ is cyclic.

Theorem 2. Let $n, k \in \mathbb{Z}^{+}$ and k be a divisor of $n.$ Then a hypergroup $(\mathbb{Z}_n, \circ'_k)$ is cyclic.

Proof. Let $n, k \in \mathbb{Z}^{+}$ such that $k \mid n.$ Then $n = mk$ for some $m \in \mathbb{Z}^{+}.$

Case 1 : $k = 1.$

There exists $1 \in \mathbb{Z}_n$ such that $1^i = \{i\},\ 1 \leq i \leq n.$ Hence

$$\bigcup_{i=1}^{n} 1^i = 1^1 \cup 1^2 \cup \cdots \cup 1^n = \{1\} \cup \{2\} \cup \cdots \cup \{n\} = \mathbb{Z}_n.$$

Case 2 : $k \in \{2, 3, \ldots, n-1\}.$

There exists $1 \in \mathbb{Z}_n$ such that $1^i = i +_n \langle m \rangle.$ Then

$$\begin{aligned}\bigcup_{i=1}^{m+1} 1^i &= 1 \cup 1^2 \cup 1^3 \cup \cdots \cup 1^{m-1} \cup 1^m \cup 1^{m+1}.\\ &= \{1\} \cup \bigcup_{i=0}^{m-1} (i +_n \langle m \rangle).\\ &= \mathbb{Z}_n.\end{aligned}$$

Case 3 : $k = n.$

We have $x \circ'_k y = x +_n y +_n \langle 1 \rangle = \mathbb{Z}_n.$ Then there exists $1 \in \mathbb{Z}_n$ such that

$$1^2 = 1 \circ'_k 1 = 1 +_n 1 +_n \langle 1 \rangle = 2 +_n \mathbb{Z}_n = \mathbb{Z}_n.$$

From all cases, there exists $1 \in \mathbb{Z}_n$ such that $\bigcup_{i\in\mathbb{Z}^+} 1^i = \mathbb{Z}_n$. This means that $(\mathbb{Z}_n, \circ'_k)$ is a cyclic hypergroup. Notice that although $(\mathbb{Z},+)$ and $(\mathbb{Z}_n,+_n)$ are cyclic groups, the hypergroup $(\mathbb{Z},\circ_k)$ is not cyclic if $k=0$ as shown in Theorem 1.

Second, to examine having property(D) of the hypergroups we find all subhypergroups of $(\mathbb{Z},\circ_k)$ and $(\mathbb{Z}_n,\circ'_k)$. Then we determine invertibilities of those subhypergroups to obtain each index of any subhypergroup.

Theorem 3. If $k=0$, then all subhypergroups of $(\mathbb{Z},\circ_k)$ are all subgroups of $\mathbb{Z}$.

Proof. Assume $k=0$. Then $x\circ_k y = x+y$ for all $x,y\in\mathbb{Z}$. Let H be a subhypergroup of $(\mathbb{Z},\circ_k)$. For any $a,b\in H$, $a+b = a\circ_k b \subseteq a\circ_k H = H$. Since $H\neq\varnothing$, there exists $c\in H$ such that $c\in c\circ_k H$. Then $c = c\circ_k h = c+h$ for some $h\in H$. Thus $0=h\in H$. Let $a\in H$. Since $0\in a\circ_k H$, there exists $h'\in H$ such that $0=a+h'$. Thus $-a=h'\in H$. This implies that H is a subgroup of $(\mathbb{Z},+)$. Conversely, suppose that H is a subgroup of $\mathbb{Z}$. Let $a\in H$. Then $H = a+H = a+H+\{0\} = a\circ_k H$. Thus H is a subhypergroup of $(\mathbb{Z},\circ_k)$. Consequently, all subhypergroups of $(\mathbb{Z},\circ_k)$ are all subgroups of $\mathbb{Z}$.

It was shown that the following subgroups of $(\mathbb{Z},+)$ are subhypergroups of $(\mathbb{Z},\circ_k)$ (Phanthawimol & Yoosomran, 2014). We also proved that these subgroups are all subhypergroups of $(\mathbb{Z},\circ_k)$ in the following theorem.

Theorem 4. Let $k\in\mathbb{Z}^+$. Then all subhypergroups of $(\mathbb{Z},\circ_k)$ are $m\mathbb{Z}$ where $m\in\mathbb{Z}\setminus\{0\}$ and $m\mid k$.

Proof. Let H be a subhypergroup of $(\mathbb{Z},\circ_k)$. Then H is a subsemigroup of $(\mathbb{Z},+)$ and there exists $h\in H\subseteq\mathbb{Z}$. Then $2h+k\mathbb{Z} = h\circ_k h\subseteq H$. If $h<0$, then $h\in H\cap\mathbb{Z}^-$ and $2h-3kh = h(2-3k) \in H\cap\mathbb{Z}^+$. If $h>0$, then $h\in H\cap\mathbb{Z}^+$ and $2h-3kh\in H\cap\mathbb{Z}^-$. If $h=0$, then $k\mathbb{Z}\subseteq H$. This implies that $H\cap\mathbb{Z}^-\neq\varnothing$ and $H\cap\mathbb{Z}^+\neq\varnothing$. By Lemma 2, $H=m\mathbb{Z}$ for some $m\in\mathbb{Z}\setminus\{0\}$. Then for any $a\in m\mathbb{Z}$, $m\mathbb{Z} = a\circ_k m\mathbb{Z} = a+m\mathbb{Z}+k\mathbb{Z}$. Because $k\in m\mathbb{Z}+k\mathbb{Z}=m\mathbb{Z}$, $m\mid k$. Therefore, the proof is complete.

Example 5. All subhypergroups of $(\mathbb{Z},\circ_4)$ are $\mathbb{Z}$, $2\mathbb{Z}$ and $4\mathbb{Z}$.

From Theorem 3 and Theorem 4, we have all subhypergroups of $(\mathbb{Z},\circ_k)$. Now, we will determine identity elements and invertibilities in order to have the index defined previously (Zhan *et al.*, 2011). It is enough to show that $(\mathbb{Z},\circ_k)$ has at least one identity element but we also find that there are countably infinite identities in the following theorem.

Theorem 6. Let $k\in\mathbb{Z}^+\cup\{0\}$. Then the set of all identity elements of $(\mathbb{Z},\circ_k)$ are $k\mathbb{Z}$.

Proof. Let $e\in k\mathbb{Z}$. Then $x=x+0\in x+k\mathbb{Z}=x+e+k\mathbb{Z}=(x\circ_k e)\cap(e\circ_k x)$ for all $x\in\mathbb{Z}$. Conversely, suppose that e is an identity element. Then $x\in(x\circ_k e)\cap(e\circ_k x)=x\circ_k e$. Thus there exists $m\in\mathbb{Z}$ such that $x=x+e+km,\ e=-km\in k\mathbb{Z}$ proving the result.

Theorem 7. Every subhypergroup of $(\mathbb{Z},\circ_k)$ is invertible.

Proof. Let H be a subhypergroup of $(\mathbb{Z},\circ_k)$ and $x,y\in\mathbb{Z}$.

Case 1 : $k=0$.

Let $H=m\mathbb{Z}$ for some $m\in\mathbb{Z}$. Assume $x\in H\circ_k y=H+y+k\mathbb{Z}=m\mathbb{Z}+y+\{0\}=m\mathbb{Z}+y$. Then $x=m\alpha_1+y$ for some $\alpha_1\in\mathbb{Z}$, so $y=x-m\alpha_1\in x+m\mathbb{Z}=m\mathbb{Z}+x+\{0\}=H\circ_k x$. Thus H is invertible on the left. Similarly, H is invertible on the right. Hence H is an invertible subhypergroup.

Case 2 : $k\in\mathbb{Z}^+$.

Let $H=m\mathbb{Z}$ for some $m\in\mathbb{Z}\setminus\{0\}$ and $m\mid k$. Assume $x\in H\circ_k y=H+y+k\mathbb{Z}=m\mathbb{Z}+y+k\mathbb{Z}$ $=y+m\mathbb{Z}$. Then $x=y+m\alpha_2$ for some $\alpha_2\in\mathbb{Z}$, so $y=x-m\alpha_2\in x+m\mathbb{Z}=m\mathbb{Z}+x+k\mathbb{Z}=H\circ_k x$. Thus H is invertible on the left. Similarly, H is invertible on the right. Hence H is an invertible subhypergroup.

Theorem 8. Let H be a subhypergroup of $(\mathbb{Z},\circ_k)$ such that $H=m\mathbb{Z}$ where $m\in\mathbb{Z}$ and $\aleph_0$ be the cardinal number of $\mathbb{Z}^+$. Then the following statements are true.

i) $[\mathbb{Z}:H]=\aleph_0$ if $k=0$ and $m=0$.

ii) $[\mathbb{Z}:H]=|m|\in\mathbb{Z}^+$ if $k=0$ and $m\in\mathbb{Z}\setminus\{0\}$.

iii) $[\mathbb{Z}:H]=|m|\in\mathbb{Z}^+$ if $k\in\mathbb{Z}^+$, $m\in\mathbb{Z}\setminus\{0\}$ and $m\mid k$.

Proof. Let $k=0$. If $m=0$, then $H=\{0\}$ and $[\mathbb{Z}:H]=\left|\{x\circ_k H\mid x\in\mathbb{Z}\}\right|=\left|\{x\mid x\in\mathbb{Z}\}\right|=\aleph_0$. If $m\in\mathbb{Z}\setminus\{0\}$, then $[\mathbb{Z}:H]=\left|\{x\circ_k H\mid x\in\mathbb{Z}\}\right|=\left|\{x+m\mathbb{Z}\mid x\in\mathbb{Z}\}\right|=|m|\in\mathbb{Z}^+$. Let $k\in\mathbb{Z}^+$. By Theorem 4, $m\mid k$. Then $[\mathbb{Z}:H]=\left|\{x\circ_k H\mid x\in\mathbb{Z}\}\right|=\left|\{x+m\mathbb{Z}+k\mathbb{Z}\mid x\in\mathbb{Z}\}\right|=\left|\{x+m\mathbb{Z}\mid x\in\mathbb{Z}\}\right|=|m|\in\mathbb{Z}^+$.

Corollary 9. Let H be a subhypergroup of $(\mathbb{Z},\circ_k)$. Then $[\mathbb{Z}:H]$ equals the number of all cosets of a subgroup H in a group $(\mathbb{Z},+)$.

Proof. It is obtained from Theorem 8.

Theorem 10. A hypergroup $(\mathbb{Z},\circ_k)$ has property(D).

Proof. Let $k=0.$ Simply note that $(\mathbb{Z},+)$ has property(D). Let $k\in\mathbb{Z}^+$ and H,K be subhypergroups of $(\mathbb{Z},\circ_k)$ with $[\mathbb{Z}:H]=[\mathbb{Z}:K].$ By Theorem 4, $H=m_1\mathbb{Z}$ and $K=m_2\mathbb{Z}$ such that $m_i\in\mathbb{Z}\setminus\{0\}$ and $m_i\mid k$ for all $i.$ By Theorem 8, $|m_1|=|m_2|.$ It follows that $H=K.$ Consequently, $(\mathbb{Z},\circ_k)$ has property(D).

Theorem 11. If $k=1,$ then all subhypergroups of $(\mathbb{Z}_n,\circ'_k)$ are all subgroups of $\mathbb{Z}_n.$

Proof. Assume $k=1.$ Then $x\circ'_k y=x+_n y$ for all $x,y\in\mathbb{Z}_n.$ Let H be a subhypergroup of $(\mathbb{Z}_n,\circ'_k).$ Let $a,b\in H,$ $a+_n b=a+_n b+_n\langle 0\rangle\subseteq H.$ Since $H\neq\varnothing,$ there exists $h'\in H$ such that $h'\in h'\circ'_k H.$ Then $h'=h'+_n h_1$ for some $h_1\in H.$ Thus $0=h_1\in H.$ Let $h\in H.$ Since $0\in h\circ'_k H,$ there exists $h''\in H$ such that $0=h+_n h''.$ Thus $-h=h''\in H.$ Hence H is a subgroup of $\mathbb{Z}_n.$ Conversely, suppose that H is a subgroup of $(\mathbb{Z}_n,+_n)$ and let $a\in H.$ Then $H=a+_n H=a+_n H+_n\langle 0\rangle=a\circ'_k H.$ Hence all subhypergroups of $(\mathbb{Z}_n,\circ'_k)$ are all subgroups of $\mathbb{Z}_n.$

It was shown that the following subgroups of $(\mathbb{Z}_n,+_n)$ are subhypergroups of $(\mathbb{Z}_n,\circ'_k)$ Phanthawimol & Yoosomran, 2014). We also proved that these subgroups are all subhypergroups of $(\mathbb{Z}_n,\circ'_k)$ in the following theorem.

Theorem 12. Let $k\in\{2,3,\ldots,n\}$ be such that $k\mid n.$ Then all subhypergroups of $(\mathbb{Z}_n,\circ'_k)$ are $\left\langle\frac{n}{v}\right\rangle$ where $v\mid n$ and $k\mid v.$

Proof. Let H be a subhypergroup of $(\mathbb{Z}_n,\circ'_k)$ and $N=\left\langle\frac{n}{k}\right\rangle.$ We will prove that $N\subseteq H$ by a contradiction.

Suppose there exists $a\in N$ but $a\notin H.$

Case 1 : $H\subset N.$

Let $b\in H.$ Then $b\circ'_k H=b+_n H+_n N=b+_n N=N\neq H,$ which contradicts that H is a subhypergroup of $(\mathbb{Z}_n,\circ'_k).$

Case 2 : $H\not\subset N.$

Case 2.1 $H\cap N\neq\varnothing.$

Let $c\in H\cap N.$ Then

$$\begin{aligned} c \circ'_k H &= \bigcup_{h \in H} c \circ'_k h, \\ &= \left(\bigcup_{h \in H-N} c \circ'_k h \right) \cup \left(\bigcup_{h \in H \cap N} c \circ'_k h \right), \\ &= \left(\bigcup_{h \in H-N} c \circ'_k h \right) \cup \left(\bigcup_{h \in H \cap N} c +_n h +_n N \right), \\ &= \left(\bigcup_{h \in H-N} c \circ'_k h \right) \cup \left(\bigcup_{h \in H \cap N} c +_n N \right), \\ &= \left(\bigcup_{h \in H-N} c \circ'_k h \right) \cup N \supseteq N. \end{aligned}$$

We have $a \in N \subseteq c \circ'_k H$ but $a \notin H$, so $c \circ'_k H \neq H$. This is a contradiction.

Case 2.2 : $H \cap N = \varnothing$.

Let $d \in H$. Then $H = d \circ'_k H = d +_n H +_n N$. Since $d \in H$, $d = d +_n d' +_n u$ for some $d' \in H$ and $u \in N$. Thus $d' = -u \in N$ whence $H \cap N \neq \varnothing$. This is a contradiction.

By Case 1 and Case 2, $N \subseteq H$.

Since H is a subhypergroup of $\mathbb{Z}_n$, $x +_n y = x +_n y +_n 0 \in x \circ'_k y \subseteq H$ for all $x, y \in H$. We have $0 \in N \subseteq H$. Let $h \in H$. Then $H = h \circ'_k H = h +_n H +_n N$. Because $0 \in N \subseteq H$ and H is closed under addition modulo n, $0 \in h +_n H +_n N = h +_n H$. Then there exists $h' \in H$ such that $0 = h +_n h'$. Thus $-h = h' \in H$. Hence H is a subgroup of $\mathbb{Z}_n$. That is, $H = \left\langle \frac{n}{v} \right\rangle$ such that $v \mid n$. Thus for any $c \in H$, $\left\langle \frac{n}{v} \right\rangle = H = c \circ'_k H = c +_n \left\langle \frac{n}{v} \right\rangle +_n \left\langle \frac{n}{k} \right\rangle$. It follows that $\frac{n}{k} \in \left\langle \frac{n}{v} \right\rangle$. Thus $k \mid v$. Consequently, $H = \left\langle \frac{n}{v} \right\rangle$ where $v \mid n$ and $k \mid v$. Therefore, the proof is complete.

Example 13. All subhypergroups of $(\mathbb{Z}_{36}, \circ'_6)$ are $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 3 \rangle$ and $\langle 6 \rangle$.

Solution. All positive divisors v of 36 are $1, 2, 3, 4, 6, 9, 12, 18$ and 36 and all divisors of v by 6 are $6, 12, 18$ and 36. Thus all subhypergroups of $(\mathbb{Z}_{36}, \circ'_6)$ are $\left\langle \frac{36}{6} \right\rangle$, $\left\langle \frac{36}{12} \right\rangle$, $\left\langle \frac{36}{18} \right\rangle$ and $\left\langle \frac{36}{36} \right\rangle$ that is, $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 3 \rangle$ and $\langle 6 \rangle$.

From previous example, we found all subhypergroups of $(\mathbb{Z}_n, \circ'_k)$ with Theorem 11 and Theorem 12. In the another way, we are able to find all subhypergroups of $(\mathbb{Z}_n, \circ'_k)$ with the lattice of subgroups of $(\mathbb{Z}_n, +_n)$. The method is to looking at the line leading to $\left\langle \frac{n}{k} \right\rangle$ in the lattice of $(\mathbb{Z}_n, +_n)$ because these are all subhypergroups of $(\mathbb{Z}_n, \circ'_k)$. From Theorem 11 and Theorem 12, we have all subhypergroups of $(\mathbb{Z}_n, \circ'_k)$. Next, we will determine identity elements and invertibilities in order to obtain the index and summarize having property(D).

Theorem 14. Let $k \in \{1, 2, ..., n\}$. Then the set of all identity elements of $(\mathbb{Z}_n, \circ'_k)$ are $\left\langle \frac{n}{k} \right\rangle$.

Proof. Let $e \in \left\langle \frac{n}{k} \right\rangle$. Then $x = x +_n 0 \in x +_n \left\langle \frac{n}{k} \right\rangle = x +_n e +_n \left\langle \frac{n}{k} \right\rangle = (x \circ'_k e) \cap (e \circ'_k x)$ for all $x \in \mathbb{Z}_n$. Conversely, suppose that e is an identity elements. Then $x \in (x \circ'_k e) \cap (e \circ'_k x) = x \circ'_k e$. Thus there exists $l \in \mathbb{Z}^+$ such that $x = x +_n e +_n l \left\langle \frac{n}{k} \right\rangle$, $e = -l \left\langle \frac{n}{k} \right\rangle \in \left\langle \frac{n}{k} \right\rangle$ proving the result.

Note that it is enough to show that $(\mathbb{Z}_n, \circ'_k)$ has at least one identity element to determine the index.

Theorem 15. Every subhypergroup of $(\mathbb{Z}_n, \circ'_k)$ is invertible.

Proof. Let $n, v \in \mathbb{Z}^+$. Let H be a subhypergroup of $(\mathbb{Z}_n, \circ'_k)$ and $x, y \in \mathbb{Z}_n$. Then $H = \left\langle \frac{n}{v} \right\rangle$ where $v \mid n$.

Case 1 : $k = 1$.

Assume $x \in H \circ'_k y = \left\langle \frac{n}{v} \right\rangle +_n y$. Then $x = u\left(\frac{n}{v}\right) +_n y$ for some $u \in \mathbb{Z}^+$. Thus $y = x - u\left(\frac{n}{v}\right) \in \left\langle \frac{n}{v} \right\rangle +_n x$ $= H \circ'_k x$. Thus H is invertible on the left. Similarly, H is invertible on the right. Hence H is an invertible subhypergroup.

Case 2 : $k \in \{2, 3, \ldots, n\}$ such that $k \mid v$ and $k \mid n$.

Note that $\frac{n}{v}\mathbb{Z}_n + \frac{n}{k}\mathbb{Z}_n = \gcd\left(\frac{n}{v}, \frac{n}{k}\right)\mathbb{Z}_n$. Let $a, b \in \mathbb{Z}^+$ such that $v = ka$ and $n = vb$. Then $\frac{n}{k} = a\left(\frac{n}{v}\right)$. This means $\frac{n}{v} \Big| \frac{n}{k}$. Assume $x \in H \circ'_k y = H +_n y +_n \left\langle \frac{n}{k} \right\rangle = \left\langle \frac{n}{v} \right\rangle +_n y +_n \left\langle \frac{n}{k} \right\rangle = \left\langle \frac{n}{v} \right\rangle +_n y$. Then $x = w\left(\frac{n}{v}\right) +_n y$ for some $w \in \mathbb{Z}^+$. Thus $y = x - w\left(\frac{n}{v}\right) \in \left\langle \frac{n}{v} \right\rangle +_n x = \left\langle \frac{n}{v} \right\rangle +_n x +_n \left\langle \frac{n}{k} \right\rangle = H \circ'_k x$, so H is invertible on the left. Similarly, H is invertible on the right. Hence H is an invertible subhypergroup.

Theorem 16. Let H be a subhypergroup of $(\mathbb{Z}_n, \circ'_k)$ such that $H = \left\langle \frac{n}{v} \right\rangle$ where $v \mid n$. Then the following statements are true.

i) $[\mathbb{Z}_n : H] = n$ if $k = 1$ and $v = 1$.

ii) $[\mathbb{Z}_n : H] = \frac{n}{v}$ if $k = 1$ and $v \in \{2,3,\ldots,n\}$.

iii) $[\mathbb{Z}_n : H] = \frac{n}{v}$ if $k, v \in \{2,3,\ldots,n\}$ such that $k \mid n$ and $k \mid v$.

Proof. Let $k = 1$. If $v = 1$, then $H = \{0\}$ and $[\mathbb{Z}_n : H] = \left|\{x \circ'_k H \mid x \in \mathbb{Z}_n\}\right| = \left|\{x \mid x \in \mathbb{Z}_n\}\right| = n.$ If $v \in \{2,3,\ldots,n\}$, then $[\mathbb{Z}_n : H] = \left|\{x \circ'_k H \mid x \in \mathbb{Z}_n\}\right| = \left|\left\{x +_n \left\langle \frac{n}{v} \right\rangle \middle| x \in \mathbb{Z}_n\right\}\right| = \frac{|\mathbb{Z}_n|}{\left|\left\langle \frac{n}{v} \right\rangle\right|} = \frac{n}{v}.$ Let $k \in \{2,3,\ldots,n\}$. By Theorem 12, $k \mid v$. By Proposition 1, $\left\langle \frac{n}{v} \right\rangle +_n \left\langle \frac{n}{k} \right\rangle = \frac{n}{v}\mathbb{Z}_n + \frac{n}{k}\mathbb{Z}_n$ $= \gcd(\frac{n}{v}, \frac{n}{k})\mathbb{Z}_n = \left\langle \frac{n}{v} \right\rangle$. Then $[\mathbb{Z}_n : H] = \left|\{x \circ'_k H \mid x \in \mathbb{Z}_n\}\right| = \left|\left\{x +_n \left\langle \frac{n}{v} \right\rangle +_n \left\langle \frac{n}{k} \right\rangle \middle| x \in \mathbb{Z}_n\right\}\right| =$ $\left|\left\{x +_n \left\langle \frac{n}{v} \right\rangle \middle| x \in \mathbb{Z}_n\right\}\right| = \frac{n}{v}.$

Corollary 17. Let H be a subhypergroup of $(\mathbb{Z}_n, \circ'_k)$. Then $[\mathbb{Z}_n : H]$ equals the number of all cosets of a subgroup H in a group $(\mathbb{Z}, +)$.

Proof. It is obtained from Theorem 16.

Theorem 18. A hypergroup $(\mathbb{Z}_n, \circ'_k)$ has property(D).

Proof. Let $k = 1$. Simply note that $(\mathbb{Z}_n, +_n)$ has property(D). Let $k \in \{2,3,\ldots,n\}$ such that $k \mid n$ and H, K be subhypergroups of $(\mathbb{Z}_n, \circ'_k)$ with $[\mathbb{Z}_n : H] = [\mathbb{Z}_n : K]$. By Theorem 12, $H = \left\langle \frac{n}{v_1} \right\rangle$ and $H = \left\langle \frac{n}{v_2} \right\rangle$ such that $v_i \mid n$ and $k \mid v_i$ for all i. By Theorem 16, $v_1 = v_2$. It follows that $H = K$. Consequently, $(\mathbb{Z}_n, \circ'_k)$ has property(D).

Discussion

This research is based on the concept of a hypergroup $(G, \circ_N)$ where N is a normal subgroup of a group G. We focus on certain hypergroups $(\mathbb{Z}, \circ_k)$ (where $k \in \mathbb{Z}^+ \cup \{0\}$) and $(\mathbb{Z}_n, \circ'_k)$ (where $n, k \in \mathbb{Z}^+$ and $k \mid n$) extended from cyclic groups $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +_n)$, respectively. It is observed that the hypergroup

$(\mathbb{Z},\circ_k)$ need not be cyclic as the group $(\mathbb{Z},+)$ for all k. However, all subhypergroups of both hypergroups are subgroups of their original cyclic groups; and the indices of subhypergroups of those hypergroups have the same results as the indices of subgroups of their original groups. Moreover, both hypergroups and their original groups have property(D). Consequently, property(D) is not a characterization of cyclic hypergroups. In our future study, we will examine the statement that if a hypergroup $(H,\circ)$ has property(D), then $(H,\circ)$ is isomorphic to (has the same structure as) $(\mathbb{Z},\circ_k)$ or $(\mathbb{Z}_n,\circ'_k)$ is true or not. To obtain a characterization of cyclic hypergroups, we will also study other properties such as property of characteristic subhypergroups and property of cyclic subhypergroups. Say a subhypergroup K of H is a characteristic subhypergroup of H if $f(K)=K$ for every automorphism f of H and note that a subhypergroup K of a cyclic hypergroup H need not be cyclic.

Conclusions

In this research, we say that a hypergroup H has property(D) if distinct subhypergroups of H have distinct indices in H. We proved that a hypergroup $(\mathbb{Z},\circ_k)$ is cyclic if $k\neq 0$ and a hypergroup $(\mathbb{Z}_n,\circ'_k)$ is cyclic if k is a positive divisor of a positive integer n where $x\circ_k y = x+y+k\mathbb{Z}$ for all $x,y\in\mathbb{Z}$ and $x\circ'_k y = x+_n y+_n \left\langle \frac{n}{k} \right\rangle$ for all $x,y\in\mathbb{Z}_n$. We also showed that both hypergroups have property(D). Hence it is not true that an arbitrary hypergroup is cyclic if and only if it has property(D). However, the indices of subhypergroups of $(\mathbb{Z},\circ_k)$ and $(\mathbb{Z}_n,\circ'_k)$ have the same results as the indices of subgroups of $(\mathbb{Z},+)$ and $(\mathbb{Z}_n,+_n)$, respectively.

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