Two Methods with the Riccati Equation to Seek Traveling Wave Solutions for the Simplified Modified Camassa-Holm Equation

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Abstract

Background and Objectives : Nonlinear evolution equations (NLEEs) are crucial in modeling numerous physical phenomena, from plasma physics to fluid mechanics. The investigation of finding solutions to nonlinear evolution equations plays an important role since those solutions can explain a variety of the problems' natural events, such as solitons, vibrations, and finite-speed propagation. There are two fundamental kinds of solutions for NPDEs: exact solutions and analytical solutions. In this work, we solve the simplified modified Camassa-Holm (SMCH) equation in the following form:

$$\theta_t + 2k\theta_x - \theta_{xxt} + s\theta^2\theta_x = 0,$$

where $s > 0,\ k$ is a real constant, and $\theta(x,t)$ represents the fluid velocity in the x-direction. We employ the traveling wave transformation to transform the simplified modified Camassa-Holm (SMCH) equation, which is a nonlinear partial differential equation, into nonlinear ordinary differential equations. Then, we solve the equation using the simple equation method with the Riccati equation and the modified extended tanh function method. Two classes of exact explicit solutions, which are in the form of generalized hyperbolic functions and generalized trigonometric functions. Additionally, the results by the simple equation method with the Riccati equation and the modified extended tanh function method are vital tools for handling further models arising in applied science and new physics. For detailed physical dynamical representation, the results can be transformed into kink waves and periodic waves. Their graphical representations are 2-D and 3-D graphs.

Methodology : Using the simple equation method with the Riccati equation and the modified extended tanh function method to solve the simplified modified Camassa-Holm (SMCH) equation. There are four main steps involved in the simple equation method with the Riccati equation:

Step 1. Wave transformation: combining the independent variables x and t into one variable, $\xi = x - \omega t.$ Then $\theta(x,t) = \theta(\xi)$ and $\xi = x - \omega t,$ where ω is the speed of a traveling wave.

Step 2. Solution assumption: suppose that the solution is in the following form: $\theta(\xi)=\sum_{i=0}^{M} a_i G^i(\xi)$ and $G'(\xi)$ conform to the following Riccati equation, $G'(\xi)=\alpha G^2(\xi)+\beta,$ where the constants α and β are nonzero.

Step 3. Finding the integer M : the positive integer M that occurs in the solution (step 2) can be estimated by taking into account the homogeneous balance between the highest-order derivative and the nonlinear terms appearing in the ordinary differential equation.

Step 4. Obtaining a solution: In order to determine $\omega, \alpha\beta,$ and $a_i,$ we must first find all terms whose coefficients are of the same order $G^i,\ i=0,1,2,3,\dots$, and then set those terms to zero. We therefore have the exact traveling wave solution.

There are five main steps involved in the modified extended tanh function method, which are as follows:

Step 1. Wave transformation: combining the independent variables x and t into one variable, $\xi = x-\omega t.$ Then $\theta(x,t)=\theta(\xi),\ \xi = x-\omega t.$

Step 2. Solution assumption: suppose that the solution in the following for

$\theta(\xi)=a_0+\sum_{i=i}^{M}\left(a_i Z^i(\xi)+b_i Z^{-i}(\xi)\right)$ and $Z'(\xi)$ conforms to the following Riccati equation,

$Z'(\xi)=\sigma+Z^2(\xi),$ in which σ is a constant.

Step 3. Finding the integer M : the positive integer M that occurs in the solution (step 2) can be estimated by taking into account the homogeneous balance between the highest-order derivative and the nonlinear terms appearing in the ordinary differential equation.

Step 4. Substitute the solution (step 2) and its derivative, as well as $Z'(\xi)=\sigma+Z^2(\xi),$ into the ordinary differential equation. Following that, by equating our $Z^i, (i=0,\pm1,\pm2,\ldots),$ coefficients to zero, we derive an algebraic system of equations that can be solved to determine the values of a_i, b_i, σ and $\omega.$

Step 5. To find the exact traveling wave solutions, substitute the values of $a_i, b_i, \sigma, \ \omega$ and from the solutions of $Z'(\xi)=\sigma+Z^2(\xi),$ into $\theta(\xi)=a_0+\sum_{i=i}^{M}\left(a_i Z^i(\xi)+b_i Z^{-i}(\xi)\right)$ as follows.

Main Results: The exact traveling wave solutions of the simplified modified Camassa-Holm (SMCH) equation by using the simple equation method with the Riccati equation, in which solutions 1-2 are shown by hyperbolic functions and solutions 3-4 are shown by trigonometric functions, are as follows:

$$\theta_{1,2}(x,t) = \pm\sqrt{\frac{6\omega\alpha\beta}{s}}\tanh\left(\sqrt{-\alpha\beta}\,(x-\omega t) - \frac{V\ln(\xi_0)}{2}\right),$$

$$\theta_{3,4}(x,t) = \pm\sqrt{\frac{-6\omega\alpha\beta}{s}}\tan\left(\sqrt{\alpha\beta}\,(x-\omega t) + \xi_0\right),$$

where $\omega = \dfrac{2k}{1-2\alpha\beta}$, $\xi_0 > 0$, $V = \pm 1$. And the exact traveling wave solutions of the simplified modified Camassa-Holm (SMCH) equation by using the modified extended tanh function method, in which solutions 5-8 are shown by hyperbolic functions and solutions 9-12 are shown by trigonometric functions, are as follows:

$$\theta_{5,6}(x,t) = \pm\sqrt{\frac{-6\omega}{s}}\left(\left(\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\,(x-\omega t)\right)\right) + \frac{\sigma}{\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\,(x-\omega t)\right)}\right),$$

$$\theta_{7,8}(x,t) = \pm\sqrt{\frac{-6\omega}{s}}\left(\left(\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\,(x-\omega t)\right)\right) + \frac{\sigma}{\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\,(x-\omega t)\right)}\right),$$

$$\theta_{9,10}(x,t) = \pm\sqrt{\frac{-6\omega}{s}}\left(\left(\sqrt{\sigma}\tan\left(\sqrt{\sigma}\,(x-\omega t)\right)\right) + \frac{\sigma}{\sqrt{\sigma}\tan\left(\sqrt{\sigma}\,(x-\omega t)\right)}\right),$$

$$\theta_{11,12}(x,t) = \pm\sqrt{\frac{-6\omega}{s}}\left(\left(\sqrt{\sigma}\cot\left(\sqrt{\sigma}\,(x-\omega t)\right)\right) + \frac{\sigma}{\sqrt{\sigma}\cot\left(\sqrt{\sigma}\,(x-\omega t)\right)}\right),$$

where $\omega = \dfrac{2k}{4\sigma+1}$.

Conclusions : The exact traveling wave solutions of the simplified modified Camassa-Holm (SMCH) equation using the simple equation method with the Riccati equation and the modified extended tanh function method. The resulting solutions are represented by hyperbolic and trigonometric functions, which can be physically converted into kink and periodic waves. The findings further the solution form of hyperbolic functions, which can be transformed into kink waves, and the solution form of trigonometric functions, which can be transformed into periodic waves. Moreover, both the simple equation method with the Riccati equation and the modified extended tanh function method rely on the Riccati equation and are straightforward to comprehend. Also, this study demonstrates that the

suggested method is appropriate and very useful for determining precise solutions to the exact traveling wave solutions to the simplified modified Camassa-Holm (SMCH) problem. The method works reliably and effectively yields accurate solutions for solitary waves.

Keywords : the simplified modified Camassa-Holm equation; the simple equation method with the Riccati equation; the modified extended tanh function method; the nonlinear partial differential equation; the traveling wave solution

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Introduction

Nonlinear evolution equations (NLEEs) are crucial in modeling numerous physical phenomena, from plasma physics to fluid mechanics. The investigation of finding solutions to nonlinear evolution equations plays an important role since those solutions can explain a variety of natural phenomena, such as solitons, vibrations, and finite-speed propagation. There are two fundamental kinds of solutions for NPDEs: exact solutions and analytical solutions. It is now easier to obtain precise solutions for NLPDEs thanks to the advancement of systematic programs like Maple and Mathematica. In recent decades, many effective methods, such as the (G'/G) - expansion method (Djilali & Ali, 2023; Phoosree & Chinviriyasit, 2021), the Kudryashov method (Kudryashov, 2020; Thadee *et al.,* 2022) the simple equation method (Sanjun & Chankaew, 2022; Chankaew *et al.,* 2023; Thadee & Phoosree, 2024), the modified simple equation method (Sheikh *et al.,* 2023), the Riccati sub-equation method (Thadee *et al.,* 2023; Phoosree *et al.,* 2024) the Riccati-Bernoulli sub-ODE method (Alharbi & Almatrafi, 2020; Sanjun *et al.,* 2024), the Sardar sub-equation method (Rehman *et al.,* 2022), the sine-Gordon method (Ananna *et al.,* 2022; Mamun *et al.,* 2024), the modified extended tanh-function method (Zahran & Khater, 2016; Sanjun *et al.,* 2024), etc.

The simplified modified Camassa-Holm (SMCH) equation (Islam *et al.,* 2023),

$$u_t + 2\alpha u_x - u_{xxt} + \beta u^2 u_x = 0, \tag{1}$$

where $\beta > 0,\ \alpha$ is a real constant, and $u(x,t)$ represents the fluid velocity in the x-direction. Although, several studies have examined the SMCH equation mathematically, few have explicitly discussed its applications in modeling real-world systems. Since the SMCH equation arises in shallow water wave theory, nonlinear optics, and plasma physics, the exact solutions obtained in this study are not only mathematically significant but also applicable to physical phenomena involving wave propagation, such as ion-acoustic waves in plasma, nonlinear fluid flows, and optical soliton dynamics. These aspects strengthen the applied relevance of the derived solutions. This equation was investigated through some methods, for instance, the exp-function method (Irshad *et al.,* 2012), the extended rational sine–cosine and sinh-cosh techniques (Onder *et al.,* 2024), the exp$\left(\phi(\eta)\right)$– expansion method (Ali *et al.,* 2016), the modified simple equation process (Islam *et al.,* 2019), the elliptic function expansion scheme (Gündogdu & Gözükızıl, 2019), the enhanced modified simple equation and the extended Kudryashov schemes (Devnath *et al.,* 2024), the (G'/G)- expansion method (Liu *et al.,* 2010), and the new auxiliary equation method (Islam *et al.,* 2023), etc.

This study distinguishes itself by the parallel application of two analytical approaches to the same equation, enabling a direct comparison of their effectiveness and resulting wave structures. Notably, the SMCH equation has not previously been analyzed using either the simple equation method with the Riccati equation or the modified extended tanh function method. This work is the first to introduce such an analysis and presents distinct solutions not reported in earlier studies. The remainder of this paper is structured as follows: In Section 2, a concise discussion of the simple equation method with the Riccati equation and the modified extended tanh function method. Section 3 is devoted to applying these methods to the SMCH problem. The graphical representation of the research findings is presented in Section 4. Section 5 compares the solutions of the SMCH equation obtained by the simple equation method with Riccati equation, the modified extended tanh function method, and the exp $\left(\phi(\eta)\right)$– expansion function expansion method. Finally, Section 6 presents the conclusion, summarizing the key findings and contributions of the study.

Methods

In this section, we present a direct method, namely the simple equation method with the Riccati equation and the modified extended tanh function method for finding the traveling wave solution to nonlinear equations. Suppose that the nonlinear partial equation, say, in two independent variables x and $t,$ is given by:

$$\mathrm{P}\left(\theta,\theta_t,\theta_x,\theta_{xx},\theta_{xt},\ldots\right)=0, \tag{2}$$

where P is, in general, a polynomial function of $\theta(x,t)$ and its arguments; the subscripts denote the partial derivatives. Start by considering combining the independent variables x and t into one variable, $\xi.$ We suppose that

$$\theta(x,t)=\theta(\xi),\ \ \xi=x-\omega t. \tag{3}$$

Where ω is the speed of the traveling wave. The wave variable (3) permits us to convert Eq. (2) into an ordinary differential equation (ODE) for $\theta = \theta(\xi)$

$$\mathrm{N}(\theta, \theta', \theta'', \theta''', \ldots) = 0, \tag{4}$$

where N is a polynomial in $\theta(\xi)$ and its derivatives in which prime indicates the derivative with respect to ξ.

The simple equation method with the Riccati equation

This method is a step following the simple equation method with the Riccati equation which the authors examined in (Nofal, 2016; Sanjun *et al.,* 2024). The main steps in this strategy are as follows:

Step 1. Start by considering Eqs. (2)-(4).

Step 2. Suppose that the solution of Eq. (4) is in the following form:

$$\theta(\xi) = \sum_{i=0}^{M} a_i G^i(\xi). \tag{5}$$

Where a_i, $i = 1, 2, 3, \ldots, M$ are constants to be determined such that $a_M \neq 0$ and $G'(\xi)$ conform to the following the Riccati equation,

$$G'(\xi) = \alpha G^2(\xi) + \beta, \tag{6}$$

where the constants α and β are nonzero. The Riccati equation of the form $G'(\xi) = \alpha G^2(\xi) + \beta$ admits different classes of solutions depending on the sign of the product $\alpha\beta$. If $\alpha\beta < 0$, the solution involves hyperbolic functions such as tanh, which are known to describe localized wave structures like kink waves. Conversely, when $\alpha\beta > 0$, the solution involves trigonometric functions such as tan, which correspond to periodic wave patterns. This classification helps to determine the physical behavior of the solutions obtained from the simple equation method. Following is an explanation of the two-case solution to Eq. (6):

Case 1: $\alpha\beta < 0$,

$$G(\xi) = -\frac{\sqrt{-\alpha\beta}}{\alpha} \tanh\left(\sqrt{-\alpha\beta}\,\xi - \frac{V \ln(\xi_0)}{2}\right), \tag{7}$$

where $\xi_0 > 0$ and $V = \pm 1$.

Case 2: $\alpha\beta > 0$,

$$G(\xi)=\frac{\sqrt{\alpha\beta}}{\alpha}\tan\left(\sqrt{\alpha\beta}\,(\xi+\xi_0)\right), \tag{8}$$

where ξ_0 is a constant.

Step 3. In Eq. (4), we apply the homogeneous balance principle between the nonlinear term and the highest-order derivative. This balancing procedure enables us to determine the positive integer value M of the solution (5).

Step 4. Insert the essential derivatives $\theta',\theta'',\ldots$ of the assumed solution into Eq. (4) for the terms that were of the same power in $G.$ By equating the coefficients G^i, $i=0,1,2,3\ldots$ to zero. We obtain $\omega,\ \alpha\beta,$ and $a_i.$ Thus, the solutions to Eq. (2) that include traveling waves are constructed.

The modified extended tanh-function method

According to the modified extended tanh-function method, this method improves upon (Zahran & Khater, 2016; Sanjun *et al.,* 2024). The main steps in this method are as follows:

Step 1. Start by considering Eqs. (2)-(4).

Step 2. Suppose that the solution of Eq. (4) is in the following form:

$$\theta(\xi)=a_0+\sum_{i=i}^{M}\left(a_iZ^i(\xi)+b_iZ^{-i}(\xi)\right). \tag{9}$$

Where a_i and b_i, $i=1,2,3,\ldots,M$ are constants to be determined such that $a_M\neq 0$ or $b_M\neq 0$ and $Z'(\xi)$ conform to the following Riccati equation,

$$Z'(\xi)=\sigma+Z^2(\xi), \tag{10}$$

in which σ is a constant. Based on the following, Eq. (6) admits a variety of solutions:

Case 1: If $\sigma<0,$ then

$$Z=-\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\,\xi\right), \tag{11}$$

or

$$Z=-\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\,\xi\right). \tag{12}$$

Case 2: If $\sigma>0,$ then

$$Z=\sqrt{\sigma}\tan\left(\sqrt{\sigma}\,\xi\right), \tag{13}$$

or

$$Z = -\sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right). \tag{14}$$

Case 3: If $\sigma = 0,$ then

$$Z = -\frac{1}{\xi}. \tag{15}$$

Step 3. In Eq. (4), we apply the homogeneous balance principle between the nonlinear term and the highest-order derivative. This balancing procedure enables us to determine the positive integer value M of the solution (9).

Step 4. Substitute Eq. (9) and its derivative as well as Eq. (10) into Eq. (4). Following that, by equating our $Z^i, (i = 0, \pm 1, \pm 2, \ldots),$ coefficients to zero, we derive an algebraic system of equations that can be solved to determine the values of $a_i, b_i, \sigma,$ and $\omega.$

Step 5. To find the exact traveling wave solutions of Eq. (2), substitute the values of $a_i, b_i, \sigma,$ and ω from the solutions of Eq. (10) into Eq. (9) as follows.

Results

In this section, we use two analytical methods to solve the SMCH equation.

$$\theta_t + 2k\theta_x - \theta_{xxt} + s\theta^2\theta_x = 0, \tag{16}$$

where $s > 0,\ k$ is a real constant. We will reduce it to an ODE using the traveling wave variable $\xi = x - \omega t.$ The substitution of the transformation into Eq. (16) leads to:

$$\left(2k - \omega\right)\theta' + \omega\theta''' + s\theta^2\theta' = 0. \tag{17}$$

Integrating Eq. (17) with the zero constant, we get:

$$\left(2k - \omega\right)\theta + \omega\theta'' + \frac{s\theta^3}{3} = 0. \tag{18}$$

The next sections employ the suggested methods to accomplish the intended outcomes.

Solutions through the simple equation method with the Riccati equation

Next, we utilized the balance approach of the highest-order derivative term θ'' with the highest nonlinear terms θ^3 in Eq. (18). Then M equals 1. We have the solution to Eq. (18) as follows:

$$\theta(\xi)=a_0+a_1G(\xi), \tag{19}$$

G satisfies Eq. (6). As a result, the θ'' and θ^3 expressions are as follows:

$$\theta''=2a_1\alpha^2G^3+2a_1\alpha\beta G, \tag{20}$$

$$\theta^3=a_0^3+3a_0^2a_1G+3a_0a_1^2G^2+a_1^3G^3. \tag{21}$$

The result of substituting Eqs. (19)–(21) into Eq. (18) is

$$\left(2ka_0-\omega a_0+\frac{sa_0^3}{3}\right)+\left(2ka_1-\omega a_1+2\omega\alpha\beta a_1+sa_0^2a_1\right)G+\left(sa_0a_1^2\right)G^2+\left(2\omega\alpha^2a_1+\frac{sa_1^3}{3}\right)G^3. \tag{22}$$

Then we set each coefficient of G^i to zero, where $i=0,1,2,3$ yields

$$G^0;\quad (2k-\omega)a_0+\frac{s}{3}a_0^3=0, \tag{23}$$

$$G^1;\quad (2k-\omega)a_1+2\omega\alpha\beta a_1+sa_0^2a=0, \tag{24}$$

$$G^2;\quad sa_0a_1^2=0, \tag{25}$$

$$G^3;\quad 2\omega\alpha^2a_1+\frac{s}{3}a_1^3=0. \tag{26}$$

When this set of mathematical equations is solved, we obtain

$$a_0=0,\ a_1=\pm\alpha\sqrt{\frac{-6\omega}{s}}\ \text{ and }\ \omega=\frac{2k}{1-2\alpha\beta}. \tag{27}$$

By Eqs. (7), (8), (27), and $\xi=x-\omega t,$ the exact traveling wave solutions of the SMCH equation are shown for two cases with an arbitrary constant ξ_0.

Case 1: $\alpha\beta<0$,

$$\theta_{1,2}(x,t)=\pm\sqrt{\frac{6\omega\alpha\beta}{s}}\tanh\left(\sqrt{-\alpha\beta}\,(x-\omega t)-\frac{V\ln(\xi_0)}{2}\right), \tag{28}$$

where $\omega = \dfrac{2k}{1-2\alpha\beta}$, $\xi_0 > 0$ and $V = \pm 1$.

Case 2: $\alpha\beta > 0$,

$$\theta_{3,4}(x,t) = \pm\sqrt{\frac{-6\omega\alpha\beta}{s}}\tan\left(\sqrt{\alpha\beta}\left(x-\omega t\right)+\xi_0\right), \tag{29}$$

where $\omega = \dfrac{2k}{1-2\alpha\beta}$ and ξ_0 is constant.

Solutions through the modified extended tanh-function method

By balancing the highest-order derivative terms θ'' with the highest nonlinear terms θ^3 in Eq. (18), the balancing number M can be defined as a positive integer, according to the process that will be defined. Therefore, M equals 1. The following is the solution to Eq. (18):

$$\theta(\xi) = a_0 + a_1 Z + b_1 Z^{-1}. \tag{30}$$

Where Z satisfies Eq. (10). Therefore, the expressions for θ'' and θ^3 expressions are as follows:

$$\theta''(\xi) = 2\sigma a_1 Z + 2a_1 Z^3 + 2\sigma^2 b_1 Z^{-3} + 2\sigma b_1 Z^{-1}, \tag{31}$$

$$\theta^3(\xi) = a_0^{\ 3} + 3a_0^2 a_1 Z + 3a_0^2 b_1 Z^{-1} + 3a_0 a_1^2 Z^2 + 6a_0 a_1 b_1 + 3a_0 b_1^2 Z^{-2} + 3a_1^2 b_1 Z + 3a_1 b_1^2 Z^{-1} + a_1^3 Z^3 + b_1^3 Z^{-3}. \tag{32}$$

The result of substituting Eqs. (30)–(32) into Eq. (18) is

$$\left(2ka_0 - \omega a_0 + \frac{sa_0^3}{3} + 2sa_0 a_1 b_1\right) + \left(2ka_1 - \omega a_1 + 2\omega\sigma a_1 + sa_0^2 a_1 + sa_1^2 b_1\right)Z + \left(2kb_1 - \omega b_1 + 2\omega\sigma b_1 + sa_0^2 b_1 + sa_1 b_1^2\right)Z^{-1}$$
$$+\left(sa_0 a_1^2\right)Z^2 + \left(sa_0 b_1^2\right)Z^{-2} + \left(2\omega a_1 + \frac{sa_1^3}{3}\right)Z^3 + \left(2\omega\sigma^2 b_1 + \frac{sb_1^3}{3}\right)Z^{-3} = 0. \tag{33}$$

Then we set each coefficient of Z^i to zero, where $i = 0, \pm 1, \pm 2, \pm 3,$ which yields

$$Z^0; \quad 2ka_0 - \omega a_0 + \frac{sa_0^3}{3} + 2sa_0 a_1 b_1 = 0, \tag{34}$$

$$Z^1; \quad 2ka_1 - \omega a_1 + 2\omega\sigma a_1 + sa_0^2 a_1 + sa_1^2 b_1 = 0, \tag{35}$$

$$Z^{-1};\quad 2kb_1-\omega b_1+2\omega\sigma b_1+sa_0^2b_1+sa_1b_1^2=0, \tag{36}$$

$$Z^{2};\quad sa_0a_1^2=0, \tag{37}$$

$$Z^{-2};\quad sa_0b_1^2=0, \tag{38}$$

$$Z^{3};\quad 2\omega a_1+\frac{sa_1^3}{3}=0, \tag{39}$$

$$Z^{-3};\quad 2\omega\sigma^2b_1+\frac{sb_1^3}{3}=0. \tag{40}$$

When these algebraic equations are resolved, we get

$$a_0=0,\ a_1=\pm\sqrt{\frac{-6\omega}{s}},\ b_1=\pm\sigma\sqrt{\frac{-6\omega}{s}}\ \text{and}\ \omega=\frac{2k}{4\sigma+1}. \tag{41}$$

By Eqs. (11)-(15), (41), and $\xi=x-\omega t,$ the exact traveling wave solutions of the SMCH equation are shown for two cases: we get

Case 1: If $\sigma<0,$ then

$$\theta_{5,6}(x,t)=\pm\sqrt{\frac{-6\omega}{s}}\left(\left(\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(x-\omega t\right)\right)\right)+\frac{\sigma}{\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(x-\omega t\right)\right)}\right), \tag{42}$$

$$\theta_{7,8}(x,t)=\pm\sqrt{\frac{-6\omega}{s}}\left(\left(\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\left(x-\omega t\right)\right)\right)+\frac{\sigma}{\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(x-\omega t\right)\right)}\right). \tag{43}$$

Where $\omega=\dfrac{2k}{4\sigma+1}.$

Case 2: If $\sigma>0,$ then

$$\theta_{9,10}(x,t)=\pm\sqrt{\frac{-6\omega}{s}}\left(\left(\sqrt{\sigma}\tan\left(\sqrt{\sigma}\left(x-\omega t\right)\right)\right)+\frac{\sigma}{\sqrt{\sigma}\tan\left(\sqrt{\sigma}\left(x-\omega t\right)\right)}\right), \tag{44}$$

$$\theta_{11,12}(x,t)=\pm\sqrt{\frac{-6\omega}{s}}\left(\left(\sqrt{\sigma}\cot\left(\sqrt{\sigma}\left(x-\omega t\right)\right)\right)+\frac{\sigma}{\sqrt{\sigma}\cot\left(\sqrt{\sigma}\left(x-\omega t\right)\right)}\right). \tag{45}$$

Where $\omega=\dfrac{2k}{4\sigma+1}.$

Discussion

The two methods' exact traveling wave solutions for the SMCH equation are in the form of hyperbolic functions and trigonometric functions. By substituting the parameters shown in Table 1 and Table 2. Table 1 displays the graph effects of Eq. (28) using the simple equation method with Riccati, which depicts the wave behaviors as kink waves, and (29), which depict the wave behaviors as periodic waves. Eqs. (42)-(43), which depicts the wave behaviors as kink waves, and Eqs. (44)-(45), which depict the wave behaviors as periodic waves, both display the graph impacts of the modified extended tanh-function method, as shown in Table 2.

It is observed that solutions expressed in hyperbolic function form generally represent kink waves, characterized by sharp, localized transitions between two distinct states. In contrast, solutions involving trigonometric functions represent periodic waves, which exhibit smooth, continuous oscillations. This distinction highlights the versatility of the SMCH equation in modeling both solitary-like and periodic behaviors, depending on the functional structure of the exact solutions.

Table 1 Parameters values of Eqs. (28) and (29)

Eqs.	Parameters	Figures	Wave effects
(28)	$\alpha=-1, \beta=2, k=-0.5, k=-2, s=2, V=1, -10\leq x,t\leq 10$	1	Kink
(29)	$\alpha=1, \beta=2, k=0.5, k=5, s=2, -10\leq x,t\leq 10$	2	periodic

Table 2 Parameters values of Eqs. (42) - (45)

Eqs.	Parameters	Figures	Wave effects
(42)	$\sigma=-1, k=2, k=4, s=5, -10\leq x,t\leq 10$	3	Kink
(43)	$\sigma=-1, k=2, k=4, s=5, -10\leq x,t\leq 10$	-	Kink
(44)	$\sigma=2, k=-2, k=-4, s=3, -10\leq x,t\leq 10$		periodic
(45)	$\sigma=2, k=-2, k=-4, s=3, -10\leq x,t\leq 10$	4	periodic

The exact solutions obtained by both methods exhibit two main wave behaviors: kink waves, represented by hyperbolic functions, and periodic waves, represented by trigonometric functions. The graphs in Figures 1–4 and parameter settings in Tables 1 and 2 confirm these characteristics. Kink waves show abrupt, localized transitions, while periodic waves display smooth oscillations. Despite their effectiveness, both methods have some limitations. They are applicable mainly when the target equation can be reduced to an ODE through a traveling wave transformation. Additionally, they assume that solutions can be expressed in terms of Riccati-type functions, which may not hold for all nonlinear equations. Comparatively, the simple equation method with the Riccati equation is more straightforward and yields concise expressions. The modified extended tanh function method, while more flexible and capable of generating a wider variety of waveforms, involves more complex algebra. Overall, both methods provide reliable and interpretable wave solutions.

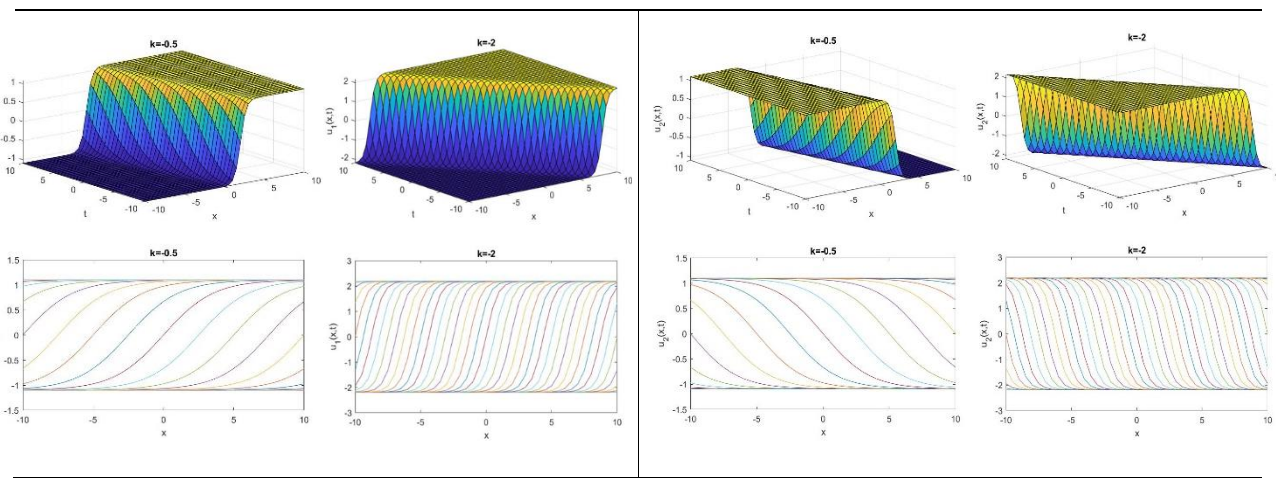

Figure 1 The kink effect in 2-D and 3-D of (28)

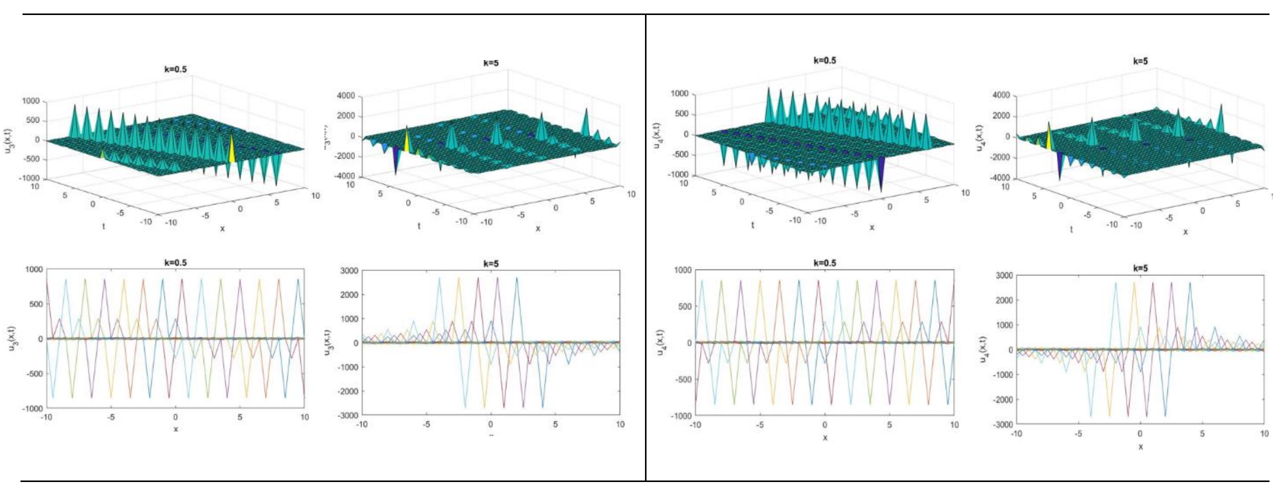

Figure 2 The periodic effect in 2-D and 3-D of (29)

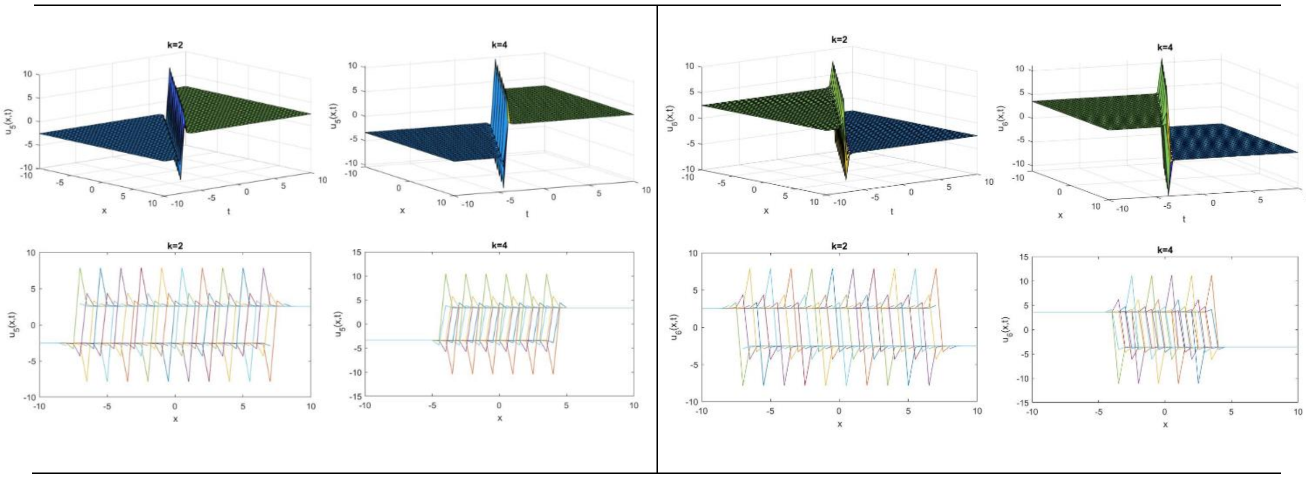

Figure 3 The kink effect in 2-D and 3-D of (41)

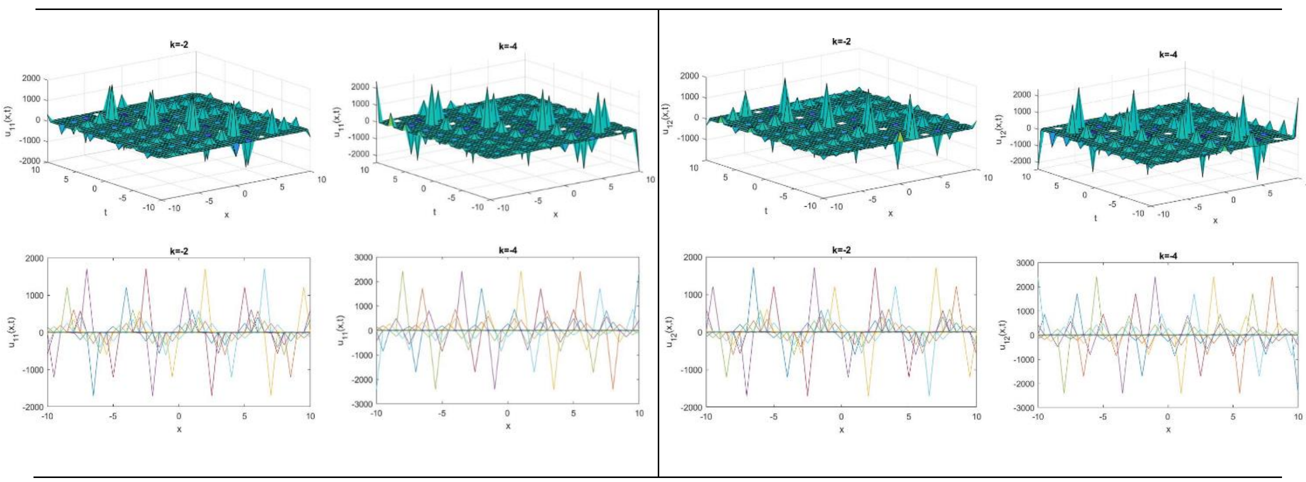

Figure 4 The periodic effect in 2-D and 3-D of (45)

Figures 1–4 illustrate the effects of wave speed ω on both kink and periodic wave solutions. In the case of the simple equation method with riccati equation, where $\omega = \dfrac{2k}{1-2\alpha\beta}$, increasing k results in sharper wave fronts and greater amplitude, particularly for kink-type waves. For the modified extended tanh function method, with $\omega = \dfrac{2k}{4\sigma+1}$, changes in k similarly affect the compactness and frequency of periodic waves. These observations confirm that parameter variations, especially in k, have a direct impact on the physical behavior of the waves.

Solutions Comparison

As shown in Tables 3 and 4, this section compares solutions of the SMCH equation from the simple equation method with Riccati and the modified extended tanh function method against those from the $\exp(\phi(\eta))-$ expansion method (Ali *et al.*, 2016).

The results demonstrate that the solutions derived from our proposed methods are more concise and structurally simpler than those from the exp expansion method. In particular, our solutions avoid lengthy fractional expressions and are expressed using standard functions such as *tanh* and *tan*, which are easier to interpret and closely related to physical wave structures. This makes the proposed methods more accessible and practical for analyzing nonlinear wave behavior.

Table 3 Solutions comparison of the SMCH equation

The $\exp(\phi(\eta))-$ expansion method	The simple equation method with Riccati equation
$u_1(\eta)=a_0-\dfrac{2\sqrt{6}\mu}{\left(-\sqrt{\lambda^2-4\mu}\tanh\left(\dfrac{\sqrt{\lambda^2-4\mu}}{2}(\eta+c_1)\right)-\lambda\right)}$, $u_2(\eta)=a_0-\dfrac{2\sqrt{6}\mu}{\left(+\sqrt{\lambda^2-4\mu}\tan\left(\dfrac{\sqrt{-\lambda^2+4\mu}}{2}(\eta+c_1)\right)-\lambda\right)}$, $u_3(\eta)=a_0-\dfrac{\sqrt{6}\mu}{\exp\left((\eta+c_1)^{\lambda}-1\right)}$, $u_4(\eta)=a_0-\dfrac{\sqrt{6}(\eta+c_1)\lambda^2}{\left(2(\eta+c_1)^{\lambda}-1\right)}$, $u_5(\eta)=a_0-\dfrac{\sqrt{6}}{(\eta+c_1)}$, where $\eta=x-Vt$.	$\theta_1(x,t)=\sqrt{\dfrac{6\omega\alpha\beta}{s}}\tanh\left(\sqrt{-\alpha\beta}(x-\omega t)-\dfrac{V\ln(\xi_0)}{2}\right)$, $\theta_2(x,t)=-\sqrt{\dfrac{6\omega\alpha\beta}{s}}\tanh\left(\sqrt{-\alpha\beta}(x-\omega t)-\dfrac{V\ln(\xi_0)}{2}\right)$, where $\omega=\dfrac{2k}{1-2\alpha\beta}$, $\xi_0>0$ and $V=\pm1.$ $\theta_3(x,t)=\sqrt{\dfrac{-6\omega\alpha\beta}{s}}\tan\left(\sqrt{\alpha\beta}(x-\omega t)+\xi_0\right)$, $\theta_4(x,t)=-\sqrt{\dfrac{-6\omega\alpha\beta}{s}}\tan\left(\sqrt{\alpha\beta}(x-\omega t)+\xi_0\right)$, where $\omega=\dfrac{2k}{1-2\alpha\beta}$ and ξ_0 is constant.

Table 4 Solutions comparison of the SMCH equation

The $\exp(\phi(\eta))$ – expansion method	The modified extended tanh function method
$u_1(\eta)=a_0-\dfrac{2\sqrt{6}\mu}{\left(-\sqrt{\lambda^2-4\mu}\tanh\left(\dfrac{\sqrt{\lambda^2-4\mu}}{2}(\eta+c_1)\right)-\lambda\right)}$, $u_2(\eta)=a_0-\dfrac{2\sqrt{6}\mu}{\left(+\sqrt{\lambda^2-4\mu}\tan\left(\dfrac{\sqrt{-\lambda^2+4\mu}}{2}(\eta+c_1)\right)-\lambda\right)}$, $u_3(\eta)=a_0-\dfrac{\sqrt{6}\mu}{\exp\left((\eta+c_1)^{\lambda}-1\right)}$, $u_4(\eta)=a_0-\dfrac{\sqrt{6}(\eta+c_1)\lambda^2}{\left(2(\eta+c_1)^{\lambda}-1\right)}$, $u_5(\eta)=a_0-\dfrac{\sqrt{6}}{(\eta+c_1)}$, where $\eta=x-Vt.$	$\theta_5(x,t)=\sqrt{\dfrac{-6\omega}{s}}\left(\left(\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}(x-\omega t)\right)\right)+\dfrac{\sigma}{\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}(x-\omega t)\right)}\right)$, $\theta_6(x,t)=-\sqrt{\dfrac{-6\omega}{s}}\left(\left(\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}(x-\omega t)\right)\right)+\dfrac{\sigma}{\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}(x-\omega t)\right)}\right)$, $\theta_7(x,t)=\sqrt{\dfrac{-6\omega}{s}}\left(\left(\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}(x-\omega t)\right)\right)+\dfrac{\sigma}{\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}(x-\omega t)\right)}\right)$, $\theta_8(x,t)=-\sqrt{\dfrac{-6\omega}{s}}\left(\left(\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}(x-\omega t)\right)\right)+\dfrac{\sigma}{\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}(x-\omega t)\right)}\right)$, $\theta_9(x,t)=\sqrt{\dfrac{-6\omega}{s}}\left(\left(\sqrt{\sigma}\tan\left(\sqrt{\sigma}(x-\omega t)\right)\right)+\dfrac{\sigma}{\sqrt{\sigma}\tan\left(\sqrt{\sigma}(x-\omega t)\right)}\right)$, $\theta_{10}(x,t)=-\sqrt{\dfrac{-6\omega}{s}}\left(\left(\sqrt{\sigma}\tan\left(\sqrt{\sigma}(x-\omega t)\right)\right)+\dfrac{\sigma}{\sqrt{\sigma}\tan\left(\sqrt{\sigma}(x-\omega t)\right)}\right)$, $\theta_{11}(x,t)=\sqrt{\dfrac{-6\omega}{s}}\left(\left(\sqrt{\sigma}\cot\left(\sqrt{\sigma}(x-\omega t)\right)\right)+\dfrac{\sigma}{\sqrt{\sigma}\cot\left(\sqrt{\sigma}(x-\omega t)\right)}\right)$, $\theta_{12}(x,t)=-\sqrt{\dfrac{-6\omega}{s}}\left(\left(\sqrt{\sigma}\cot\left(\sqrt{\sigma}(x-\omega t)\right)\right)+\dfrac{\sigma}{\sqrt{\sigma}\cot\left(\sqrt{\sigma}(x-\omega t)\right)}\right)$, where $\omega=\dfrac{2k}{4\sigma+1}.$

Conclusions

The simple equation method with the Riccati equation and the modified extended tanh function method presented in this article have been successfully implemented to find exact traveling wave solutions for the SMCH equation. The solutions are expressed in trigonometric and hyperbolic forms. Both methods provide useful solutions to the SMCH problem, demonstrating that they are practical and effective analytical approaches. Furthermore, we presented 2-D and 3-D plots of the SMCH equation solutions obtained by the simple equation method with the Riccati equation (Figures 1–2) and the modified extended tanh function method (Figures 3–4), where all graphs depict kink and periodic waves.

Throughout this work, the parallel application of two analytical methods allowed for a direct comparison of their respective solution structures and effectiveness, providing deeper insight into the wave dynamics governed by the SMCH equation. The obtained solutions are not only mathematically significant but also have potential applications in modeling physical phenomena such as shallow water waves, nonlinear optics, and plasma physics.

For future work, it would be worthwhile to explore additional analytical techniques that may yield a wider variety of exact solutions, thereby revealing more diverse wave structures. Employing alternative methods could provide a deeper understanding and richer solution sets for the SMCH equation.

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