BSJ
BURAPHA SCIENCE JOURNAL
ISSN 2985-0983 (online)

วารสารวิทยาศาสตร์บูรพา ปีที่ 30 (ฉบับที่ 3) กันยายน – ธันวาคม พ.ศ. 2568
BURAPHA SCIENCE JOURNAL Volume 30 (No.3) September – December 2025 บทความวิจัย

# On the Diophantine Equation $p^x + n^y = z^2$ where $p$ is an Odd Prime and $n$ is a Non-Negative Integer with $n \equiv 7 \pmod{12}$ and $\gcd(n, p) = 1$

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Received : 15 July 2025, Received in revised form : 17 October 2025, Accepted : 19 October 2025

Available online : 5 November 2025

## Abstract

**Background and Objectives :** Diophantine equations, which seek integer solutions to polynomial equations, have long been a fundamental and extensively studied topic in number theory. Among these, exponential Diophantine equations, where variables appear as exponents, are particularly challenging due to their nonlinear nature and profound connections to classical conjectures such as Catalan's Conjecture. This conjecture, proven by Mihailescu in 2004, states that the equation $x^a - y^b = 1$, where $a, b, x$ and $y$ are integers with min $\{a, b, x, y\} > 1$ has the unique positive integer solution $(x, y, a, b) = (3,2,2,3)$. This study focuses on the exponential Diophantine equation $p^x + n^y = z^2$, where $p$ is an odd prime, and $n, x, y, z$ are non-negative integers. The primary objective is to determine all possible integer solutions of this equation under various conditions, with particular emphasis on modular restrictions imposed on the parameter $n$. The research investigates the interplay between the prime base $p$ and the parameter $n$ in determining the existence of solutions. Previous works have contributed foundational insights into related problems. Notably, Nagell (1948) proved the finiteness of solutions to the Lebesgue-Nagell equation $x^2 + D = y^n$, for fixed integers $D$ and $n$. Tijdeman (1976) extended these results using Baker's theory of linear forms in logarithms to show the finiteness of positive integer solutions to exponential Diophantine equations of the form $a^x + b^y = c$, for fixed integers $a, b, c$. While these studies have illuminated important aspects of the problem, a comprehensive understanding of solutions under specific modular constraints on $n$ remains incomplete. Therefore, this research conducts a detailed analysis of the equation under the modular condition $n \equiv 7 \pmod{12}$. This is achieved by employing classical number theory tools such as the theory of quadratic residues and the Legendre symbol to rigorously restrict the possible solution values. The methodology involves transforming the equation into forms amenable to modular arithmetic analysis and factorization, with the goal of establishing necessary and sufficient conditions for the existence of non-negative integer solutions. Analyzing these constraints will allow a complete characterization of all possible solution sets and demonstrate the absence of solutions outside

these conditions. The results are expected to deepen the theoretical understanding of exponential Diophantine equations involving prime powers and perfect squares under modular constraints, extend classical results, and provide a framework for investigating more complex cases in future research.

**Methodology** : The investigation begins by transforming the equation $p^x + n^y = z^2$ where $p$ is an odd prime, and $n, x, y, z$ are non-negative integers, into forms that are amenable to analysis using modular arithmetic and the properties of quadratic residues. A key insight involves the application of quadratic residues modulo 12, which impose stringent restrictions on the possible values of $n$, thereby significantly influencing the solvability of the equation. Specifically, the congruence condition $n \equiv 7 (\mathrm{mod}\ 12)$ emerges naturally from residue computations and serves as a critical criterion for filtering candidate solutions. The research further explores the relationship between the given Diophantine equation and the generalized Pell equation of the form $x^2 - Dy^2 = 1$, which is known to have infinitely many integer solutions under certain conditions. By establishing this connection, the study relates the growth behavior of the original equation's solutions to those of Pell-type equations. Techniques derived from the theory of linear forms in logarithms, inspired by the foundational work of Tijdeman and Baker, are employed to establish explicit upper bounds on the exponents $x$ and $y$ in terms of $p$ and $n$. Throughout the analysis, modular constraints and the greatest common divisor condition $\gcd(n, p) = 1$ are examined systematically. The proof strategy integrates modular arithmetic with prime factorization methods and classical analytic number theory to derive necessary and sufficient conditions for the existence of non-negative integer solutions to the equation.

**Main Results** : The research shows that the Diophantine equation $p^x + n^y = z^2$ admits non-negative integer solutions only under highly restrictive circumstances. Specifically, the equation has solutions precisely when $p = 3$ and $\gcd(n, p) = 1$, with the complete solution set given by $(p, n, x, y, z) \in \{(3, n, 1, 0, 2) \cup (3, 2 \cdot 3^s + 1, 2s, 1, 3^s + 1) | s \in \mathbb{Z} \geq 0\}$, where $s$ is a non-negative integer parameter that generates an infinite family of solutions directly related to Pell-type equations. This characterization reveals an intricate structural link between the original exponential Diophantine equation and quadratic forms. Additionally, the modular condition $n \equiv 7 (\mathrm{mod}\ 12)$ is proven to be both necessary and natural for solutions to exist. The analysis confirms that for any other values of $p$ or $n$ not satisfying these conditions, the equation has no non-negative integer solutions. This result not only aligns with but also extends existing theorems on the rarity of solutions to exponential Diophantine equations. It demonstrates how quadratic residue conditions can tightly constrain possible solutions, providing a deep understanding of the interplay between the arithmetic properties of $p$ and the modular behavior of $n$. Furthermore, the results establish a bridge to classical results on Pell equations by showing that the infinite familie of solutions correspond precisely to sequences generated by fundamental solutions to related Pell-type equations.

**Conclusions** : This study provides a complete characterization of non-negative integer solutions to the Diophantine equation $p^x + n^y = z^2$ for odd primes $p$, demonstrating that solutions exist only under the specific modular condition $n \equiv 7 (\mathrm{mod}\ 12)$ and with $p = 3$ satisfying $\gcd(n, p) = 1$. The explicit forms of the solutions reveal a unique structure rooted in connections to Pell-type equations and quadratic residue theory. These findings deepen our understanding of exponential Diophantine equations involving prime powers and perfect squares under modular constraints. The research contributes new insight to the field by elucidating the conditions under which such equations are solvable, laying a foundation for future work on more complex equations involving higher powers or additional variables. Potential applications of the methodological advances include algorithmic number theory, cryptography, and computational mathematics, where understanding the interaction between primes, exponents, and perfect powers is crucial. By combining classical and modern techniques, this research not only extends the known results but also provides a template for studying other exponential Diophantine equations with similar structural properties, opening avenues for further exploration in both pure and applied mathematics.

**Keywords** : Diophantine equation ; Legendre symbol; quadratic residue; number theory

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## Introduction

A Diophantine equation is an equation with one or more unknown variables that has solutions restricted to non-negative integers. These types of equations are an important topic in number theory, especially exponential equations where variables appear as exponents. Such equations are highly complex due to their nonlinear nature and are connected to deep theoretical concepts, such as Catalan's Conjecture, which was proven by Mihailescu in 2004 to have the unique positive integer solution $a^p - b^q = 1$, where $(a, p, b, q) = (3,2,2,3)$. A classical result by Nagell (1948) proved that the Lebesgue–Nagell equation $x^2 + D = y^n$ has only finitely many solutions for fixed constants $D$ and $n$. Later, Tijdeman (1976) extended this by applying Baker's theory of linear forms in logarithms to show that exponential Diophantine equations of the form. $x^p - y^q = c$ with $p, q \geq 2$ have finitely many solutions. In recent decades, mathematicians have extensively studied various forms of Diophantine equations. For example, Sroysang (2013) investigated the equation $5^x + 7^y = z^2$, and found no solutions in non-negative integers. In 2014, Sroysang (2014) also studied $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$, proving that neither has non-negative integer solutions. Similarly, Burshtein (2020) analyzed the equation $7^x + 11^y = z^2$ and confirmed there are no positive integer solution. Orosram & Comemuang (2020) found the unique solution $(x, y, z) = (1,0,3)$ for $8^x + n^y = z^2$, which is the only non-negative integer solution. Viriyapong & Viriyapong (2021) studied $n^x + 13^y = z^2$, where $n$ is

a positive integer with $n \equiv 2(\mathrm{mod}\ 39)$ and $n+1$ is not a square number, and found a unique solution $(n,x,y,z) = (2,3,0,3)$. Moonchaisook (2022) examined $p^x + (2p-1)^y = z^2$ where $p$ and $2p-1$ are prime numbers, finding no positive integer solutions. Pakapongpun & Chattae (2022) extended the study of $p^x + 7^y = z^2$, identifying some specific solutions. Jantoy & Intep (2024) showed that for many cases of $t^x + (t+3k)^y = z^2$, there are no non-negative integer solutions when $t$ and $k$ lie within certain ranges. Tadee (2025) investigated cubic exponential Diophantine equations of the form $13^x + a^y = z^3$, Identifying conditions for both existence and non-existence of non-negative integer solutions.

However, Diophantine equation of the form

$$p^x + n^y = z^2 \tag{1}$$

where $p$ is an odd prime, $n \equiv 7(\mathrm{mod}\ 12)$, and $x, y, z$ are non-negative integers, have not been thoroughly analyzed to determine the necessary and sufficient conditions for the existence of solutions. This research aims to systematically study such equations by transforming them into forms suitable for modular arithmetic and quadratic residue analysis, focusing on identifying possible values of $n, x, y, z$ under modular constraints. The study employs elementary number theory techniques, such as properties of quadratic residues and the Legendre symbol, to exclude impossible cases and confirm valid solutions. This approach enables a complete and systematic characterization of all possible solutions.

## Methodology

This section presents the key definitions, lemmas, and theorems that are used in the proofs of the main theorems in this paper. The proofs of these results are not provided here, readers who are interested may refer to the cited references for further details.

**Definition 1.** (Burton, 2010)

Let $p$ be an odd prime and let $a$ be an integer such that $\gcd(a,p) = 1$. If the congruence $x^2 \equiv a(\mathrm{mod}\ p)$ has a solution, then $a$ is called a quadratic residue modulo $p$; otherwise, $a$ is called a quadratic nonresidue modulo $p$.

**Definition 2.** (Burton, 2010)

Let $p$ be an odd prime number and $a$ be an integer such that $\gcd(a,p) = 1$.

The Legendre symbol $\left(\frac{a}{p}\right)$ is defined as follows.

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0\ (\mathrm{mod}\ p), \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p, \\ 0 & \text{if } a \equiv 0(\mathrm{mod}\ p). \end{cases}$$

**Theorem 1.** (Burton, 2010)

If $p$ is a prime number that is an odd integer, then

$$\left(\frac{2}{p}\right) = \begin{cases} 1, & p \equiv 1(\text{mod } 8) \text{ or } p \equiv 7(\text{mod } 8), \\ -1, & p \equiv 3(\text{mod } 8) \text{ or } p \equiv 5(\text{mod } 8). \end{cases}$$

**Proposition 1.** If $x$ is an integer, then $x^2 \equiv 0, 1(\text{mod } 4)$.

**Proof.** See Thongnak *et al*. (2022).

**Lemma 1.** Let $p$ be a prime number. Then the Diophantine equation $p^x + 1 = z^2$ has exactly two solutions in non-negative integers $(p, x, z)$, namely $(2,3,3)$ and $(3,1,2)$.

In particular, if $p$ is an odd prime, the unique solution is $(3,1,2)$.

**Proof.** We distinguish two cases.

**Case 1.** $p = 2$.

The equation becomes $2^x = z^2 - 1 = (z-1)(z+1)$. Since $z$ must be odd,

$\gcd(z-1)(z+1) = 2$. Thus both factors are powers of 2, say $z - 1 = 2^m, z + 1 = 2^n$ with $n > m \geq 1$.

Subtraction gives $2^n - 2^m = 2$. which means that $2^m(2^{n-m} - 1) = 2$. The only solution is $m = 1$ and $n - m = 1$, which yields $b = 2$, hence $z = 3$ and $x = 3$. Therefore the solution is $(2, 3, 3)$.

**Case 2.** $p > 3$.

Then we have $p^x = (z-1)(z+1)$ with $\gcd(z-1, z+1) = 1$. Thus each factor is a power of $p$, say $z - 1 = p^a$ and $z + 1 = p^b$ with $b > a \geq 0$. Subtracting gives $p^b - p^a = 2$, which implies that $p^a(p^{b-a} - 1) = 2$.

Since $p$ is odd, this equation forces $a = 0$ and $p^b - 1 = 2$. Hence $p^b = 3$, so $p = 3$ and $b = 1$. Consequently, $z = 2$ and $x = 1$. Therefore the unique solution in this case is $(3,1,2)$.

This completes the proof.

**Lemma 2.** Let $y, z$ and $n$ be non-negative integers with $n \equiv 7(\text{mod } 12)$.

Then the Diophantine equation $1 + n^y = z^2$ has no solution.

**Proof.** Assume that $(n, y, z)$ is a solution in non-negative integers of the equation $1 + n^y = z^2$.

Suppose that $n \equiv 7(\text{mod } 12)$. Then it follows that $n \equiv 1(\text{mod } 3)$, we have $n^y \equiv 1(\text{mod } 3)$..

Then $1 + n^y \equiv 2(\text{mod } 3)$. Thus, $z^2 \equiv 2(\text{mod } 3)$, which is a contradiction, because 2 is a quadratic non-residue modulo 3. By Theorem 1, $\left(\frac{2}{3}\right) = -1$, and hence, there are no integers $z$ satisfying the equation in this case.

Therefore, there is no non-negative integer solution.

**Lemma 3.** Let $A \in \mathbb{Z}$ and $k \geq 2$.

Then $$(12A+7)^k \equiv \begin{cases} 1 \pmod{12}, & \text{if } k \text{ is even}, \\ 7 \pmod{12}, & \text{if } k \text{ is odd}. \end{cases}$$

**Proof.** Observe that $12A + 7 \equiv 7(mod\ 12)$.

Hence $(12A+7)^k \equiv 7^k \pmod{12}$, so it suffices to determine the residue of $7^k$ modulo 12.

We proceed by mathematical induction on $k \geq 0$.

When $k = 0$, clearly $7^0 = 1 \equiv 1 \pmod{12}$ (even exponent).

When $k = 1$, we have $7^1 = 7 \equiv 7 \pmod{12}$ (odd exponent).

Thus, the claim holds for $k = 0$ and $k = 1$.

Assume now that for some $n \geq 1$, the statement holds

$$7^n \equiv \begin{cases} 1 \pmod{12}, & n \text{ is even}, \\ 7 \pmod{12}, & n \text{ is odd}. \end{cases}$$

We must show that the result also holds for $n + 1$. We compute $7^{n+1} \equiv 7 \cdot 7^n$.

If $n$ is odd, then by the induction hypothesis, $7^n \equiv 7(\mathrm{mod}\ 12)$. Thus, $7^{n+1} \equiv 7 \cdot 7 = 49 \equiv 1 \pmod{12}$.

If $n$ is even, then $7^n \equiv 1(\mathrm{mod}\ 12)$. So $7^{n+1} \equiv 7 \pmod{12}$. Hence, the statement holds for $n$.

Thus the statement holds for $n + 1$ whenever it holds for $n$.

By the principle of mathematical induction, the claim is established for all $k \geq 0$.

**Lemma 4.** Let $M \in \mathbb{Z}$ and $n \in \mathbb{Z} \geq 0$.

Then $$(12M+3)^n \equiv \begin{cases} 1(\mathrm{mod}\ 12), \text{if } n = 0, \\ 3(\mathrm{mod}\ 12), \text{if } n \text{ is odd}, \\ 9(\mathrm{mod}\ 12), \text{if } n \text{ is even}. \end{cases}$$

**Proof.** Since $12M + 3 \equiv 3(\mathrm{mod}\ 12)$, it suffices to evaluate $3^n(\mathrm{mod}\ 12)$. By the same inductive argument as in Lemma 3, we obtain $3^n \equiv 3(\mathrm{mod}\ 12)$, if $n$ is odd $3^n \equiv 9(\mathrm{mod}\ 12)$, if $n$ is even, and $3^0 \equiv 1(\mathrm{mod}\ 12)$. This completes the proof.

**Lemma 5.** Let $s \in \mathbb{Z} \geq 1$ and $n, y \in \mathbb{Z} > 0$ satisfy

$n^y = 2 \cdot 3^s + 1$ and $n \equiv 7(\mathrm{mod}\ 12)$. Then $y = 1$.

**Proof.** Suppose, for contradiction, that $y > 1$. Since $n \equiv 7(\mathrm{mod}\ 12)$, write $n \equiv 12A + 7$ with $A \in \mathbb{Z}$. By Lemma 3, for every integer $k \geq 1$ we have $(12A + 7)^k \equiv 7(\mathrm{mod}\ 12)$ when $k$ is odd and $(12A + 7)^k \equiv 1(\mathrm{mod}\ 12)$ when $k$ is even. Because $n^y = 2 \cdot 3^s + 1 \equiv 7(\mathrm{mod}\ 12)$, Lemma 3 forces $y$ to be odd, in particular $y \geq 3$. Subtracting 1 from both sides of $n^y = 2 \cdot 3^s + 1$ yields $n^y - 1 = 2 \cdot 3^s$. Using the difference-of-powers identity,

$$n^y - 1 = (n-1)(n^{y-1} + n^{y-2} + \ldots + 1).$$

Define

$$S := \sum_{i=0}^{y-1} n^i = n^{y-1} + n^{y-2} + \ldots + 1.$$

Since $n \equiv 7 (\mathrm{mod}\ 12)$, the integer $n$ is odd; and as $y$ is odd, each summand $n^i$ is odd and the number of summands is odd, hence $S$ is odd. From the defining relation $n^y = 2 \cdot 3^s + 1$ we obtain

$$n^y - 1 = (n-1)S = 2 \cdot 3^s.$$

From $(\mathrm{n}-1)\mathrm{S} = 2 \cdot 3^s$, every prime divisor of $\mathrm{n}-1$ and of S belongs to $\{2,3\}$. Since S is odd, all its prime divisors are equal to 3. Moreover, $\mathrm{n} \equiv 7 (\mathrm{mod}\ 12)$ implies $n \equiv 1 (\mathrm{mod}\ 3)$, hence $3 | (n-1)$.

Write $n - 1 = 2^a 3^t, \quad a, t \in \mathbb{Z} \geq 1$. Then $S = \frac{2 \cdot 3^s}{2^a 3^t} = 2^{1-a} 3^{s-t}$.

Because $S$ is odd, we must have $a = 1$; therefore $S = 3^{s-t}$ with $t \geq 1$. In particular, $S \leq 3^{s-1} \leq 3^s$.

On the other hand, since $y \geq 3$,

$$S = n^{y-1} + n^{y-2} + \ldots + 1 \geq 1 + n + n^2 > n^2.$$

Using $n = 2 \cdot 3^s + 1$, we compute

$$n^2 = (2 \cdot 3^s + 1)^2 = 4 \cdot 3^{2s} + 4 \cdot 3^s + 1 > 4 \cdot 3^{2s},$$

Hence $S > 4 \cdot 3^{2s}$. Combining this with the previous bound $S \leq 3^s$ gives $\mathrm{S} \leq 3^s$ and $S > 4 \cdot 3^{2s}$,

Which is impossible, since

$$\frac{4 \cdot 3^{2s}}{3^s} = 4 \cdot 3^s \geq 12 \qquad (s \geq 1).$$

This contradiction completes the proof.

## Results

The study of finding the solution of the Diophantine equation $p^x + n^y = z^2$ for $n, x, y$ and $z$ are non-negative integer found that this equation has non-negative integer solutions $(p, n, x, y, z)$ are non-negative integer that is $(p, n, x, y, z) \in \{(3, n, 1, 0, 2) \cup (3, 2 \cdot 3^s + 1, 2s, 1, 3^s + 1)\}$ for $s$ is a non-negative integer where $n \equiv 7 (\mathrm{mod}\ 12)$ and $p$ is an odd prime and $\gcd(n, p) = 1$. We can present the proof of this theory as follows.

**Theorem 2.** Let $p$ be an odd prime number and let $n$ and $y$ be non-negative integers such that $n \equiv 7 (\mathrm{mod}\ 12)$ and $\gcd(n, p) = 1$. Then, the Diophantine equation $p^x + n^y = z^2$ has a solution in non-negative $(x, y, z)$ of the form $(p, n, x, y, z) \in \{(3, n, 1, 0, 2) \cup (3, 2 \cdot 3^s + 1, 2s, 1, 3^s + 1)\}$, where $s$ is a non-negative integer.

**Proof.** The Diophantine equation $p^x + n^y = z^2$ is divided into 4 cases as follows.

**Case 1** For $x = 0$ and $y = 0$, so $z^2 = 2$ is impossible because z is an integer.

**Case 2** Let $x = 0$ and $y \geq 1$, so that the equation becomes $1 + n^y = z^2$. According to Lemma 2, it can be concluded that the equation $1 + n^y = z^2$ has no solution in non-negative integers.

**Case 3** For the case $y = 0$ and $x \geq 1$, the equation $p^x + 1 = z^2$,

has by Lemma 1, a solution in non-negative integers given by $(p, x, y, z) = (3,1,0,2)$.

**Case 4** Let $x \geq 1$ and $y \geq 1$. Since $p$ is an odd prime, and $n \equiv 1(\text{mod } 3)$ and $n \equiv -1(\text{mod } 4)$,

we consider the following three cases.

**Case 4.1** Let $p = 3$.

From equation (1), we obtain $3^x + n^y = z^2 \equiv (-1)^x + (-1)^y(\text{mod } 4)$, since $z^2$ is an even number and satisfies $z^2 \equiv 0(\text{mod } 4)$. Therefore we need to consider the following two subcases for the parities of $x$ and $y$.

**Subcase 4.1.1** Let $x$ be even and let $y$ be odd.

Suppose $x = 2s \geq 1$, where $s$ are non-negative integers. So that $3^x = (3^s)^2$.

We consider the Diophantine $(3^s)^2 + n^y = z^2$.

Observe that we can rearrange this equation as a difference of squares $n^y = z^2 - (3^s)^2$.

which factors as $n^y = (z + 3^s)(z - 3^s)$.

Let $y = v + \gamma$, $n^v = z + 3^s$ and $n^\gamma = z - 3^s$ for some non-negative integer $v, \gamma$ and $v > \gamma \geq 0$,

which makes $n^\gamma[n^{v-\gamma} - 1] = 2 \cdot 3^s$. Since $v > \gamma$, it follows that $n^{v-\gamma} - 1$ is an integer.

Thus $n^\gamma | 2 \cdot 3^s$ . Given that $\gcd(3, n) = 1$, it follows that $n^\gamma | 2$. Since $n \equiv 7(\text{mod } 12)$, we have $n^\gamma = 1$ ,that is

$\gamma = 0, z = 3^s + 1$. So we have $n^v - 1 = 2 \cdot 3^s$, that is $n^y = 2 \cdot 3^s + 1$.

From the equation $n^y = 2 \cdot 3^s + 1$, we aim to determine a value of $n$ such that the condition $n \equiv 7(\text{mod } 12)$ is satisfied, as required by Lemma 3.

According to Lemma 3, if $n \equiv 7(\text{mod } 12)$, then

$$n^k \equiv \begin{cases} 1 \ (\text{mod } 12), \text{if } k \text{ is even}, \\ 7 \ (\text{mod } 12), \text{if } k \text{ is odd}. \end{cases}$$

Since $y$ is odd, we have $n^y \equiv 7(\text{mod } 12)$. Thus, $n^y - 1 \equiv 6(\text{mod } 12)$.

From the relation $n^y = 2 \cdot 3^s + 1$, we obtain $2 \cdot 3^s + 1 \equiv 7(\text{mod } 12)$.

Because $3^s \equiv 3(\text{mod } 12)$ when $s$ is odd and $3^s \equiv 9(\text{mod } 12)$ when $s$ is even, the congruence holds for all integers $s \geq 1$. Thus the modular condition imposes no further restriction on $s$.

Finally, by applying Lemma 5, we deduce that $y = 1, n = 2 \cdot 3^s + 1, z = 3^s + 1$, which is fully consistent with the congruence in Lemma 3.

From $n = 2 \cdot 3^s + 1$, let $n = a \cdot b$, where $a, b \in \mathbb{Z}^+$and $a \leq b$. Consider the following two cases.

1) $n$ is a prime number.

We set $a = 1$. Therefore $b = n = 2 \cdot 3^s + 1, y = 1, z = 3^s + 1$.

Thus $(p,n,x,y,z) = (3, 2\cdot 3^s+1, 2s, 1, 3^s+1)$.

2) $n$ is a composite number.

Assume $n = a\cdot b$ with $1 < a \le b$, then $ab = 2\cdot 3^s+1,\ y = 1$.

We then obtain, $z^2 = 3^{2s}+n = 3^{2s}+(2\cdot 3^s+1) = (3^s+1)^2 = 3^{2s}+ab = 3^{2s}+n$ for the given $s \ge 1$.

Hence, subcase 4.1.1 has a solution $(p,n,x,y,z)$ which are non-negative integer $(3, ab, 2s, 1, 3^s+1)$.

**subcase 4.1.2** Let $x$ is odd and y is even.

Suppose $x = 2s+1$ and $y = 2f$, where $f$ and $s$ are non-negative integers.

Then, we have the Diophantine equation $3^x + (n^f)^2 = z^2$.

This can be rewritten as a difference of squares, $3^x = z^2 - (n^f)^2 = (z+n^f)(z-n^f)$.

Let $3^{x-h} = z+n^f$, and $3^h = z-n^f$, where $h$ is anon-negative integer such that $x > h$.

Then, $3^h[3^{x-2h}-1] = 2\cdot n^f$. We let $h$ be a non-negative integer satisfying the condition $x-2h \ge 0$.

Under this assumption, the term $3^{x-2h}-1$ is an integer, and it follows that $3^h \mid 2\cdot n^f$.

Since $\gcd(3,n) = 1$, by Lemma 3 we have $2\cdot n^f + 1 \equiv 3 (\mathrm{mod}\ 12)$.

From Lemma 4, since $3^x \equiv 3 (\mathrm{mod}\ 12)$, we get $3^x = 3^{2s+1} \equiv 3 (\mathrm{mod}\ 12)$, implying that $3^{2s} \equiv 1 (\mathrm{mod}\ 12)$.

Thus $s = 0$, and hence $x = 1$. Since $2\cdot n^f \equiv 2 (\mathrm{mod}\ 12)$, we conclude that $n^f \equiv 1 (\mathrm{mod}\ 12)$.

Hence $n^f = 1$, which gives $f = 0$. and $y = 0$. Therefore, there exists a solution in this case $(p,x,y,z) = (3,1,0,2)$. However, under the condition $y \ge 1$, we must have $f \ge 1$, and thus the assumption $f = 0$ is no longer valid. Consequently, the conclusion $(p,x,y,z) = (3,1,0,2)$ does not satisfy the required condition $y \ge 1$, and must be excluded from the set of valid solutions.

**Case 4.2** Let $p \equiv 1 (\mathrm{mod}\ 3)$.

From equation (1), it can be concluded that $z^2 \equiv 2 (\mathrm{mod}\ 3)$.

By Theorem 1, $\left(\frac{2}{3}\right) = -1$. Therefore, in this case, there is no solution.

**Case 4.3** Let $p \equiv 2\ (\mathrm{mod}\ 3)$. It follows immediately that $p > 3$.

We divide the consideration into three subcases on the parities of $x$ and $y$.

**Subcase 4.3.1** $x$ is an even number and $y \ge 1$ such that $p^x \equiv 1 (\mathrm{mod}\ 3)$ and $n \equiv 1 (\mathrm{mod}\ 3)$.

From equation (1), it can be concluded that $z^2 \equiv 2 (\mathrm{mod}\ 3)$. By Theorem 1, $\left(\frac{2}{3}\right) = -1$.

Therefore, in this case, there is no solution.

**Subcase 4.3.2** Let $x$ be an odd number and $y$ an even number, Set $y = 2f$ for some non-negative integers $f$. From equation (1), we have $p^x = (z-n^f)(z+n^f)$. Which implies there exists a non-negative

integer $r$ such that $p^r = z - n^f$ and $p^{x-r} = z + n^f$. Since $z + n^f > z - n^f$, we must have $x - r > r$, that is $x > 2r$. Then we find $(z + n^f) - (z - n^f) = p^{x-r} - p^r$, so that $2n^f = p^r(p^{x-2r} - 1)$.. Because $x > 2r$, it follows that $p^{x-2r} - 1$ is an integer, which leads to $p^r | 2n^f$. Now, since $\gcd(p, n) = 1$, it follows that $p^r | 2$. Therefore $p^r = 2^0 = 1$, which gives $r = 0$. Thus, $p^x = 2n^f + 1$. However, since $p^x \equiv 2 (\mathrm{mod}\ 3)$.

Hence, no solution exists in this case.

Subcase 4.3.3 Let $x$ be an odd number and $y$ be an odd number,

Assume that $x = 2s + 1$ and $y = 2f + 1$, where $s, f \in \mathbb{Z} \geq 0$.

Since $p \equiv 2 (\mathrm{mod}\ 3)$, it follows that:

$$p^x = p^{2s+1} \equiv 2 (\mathrm{mod}\ 3), \text{ and } n^y = n^{2f+1} \equiv 1 (\mathrm{mod}\ 3).$$

Bstituting into the original Diophantine equation:

$$p^x - 1 = z^2 - n^{2f+1} - 1. \tag{2}$$

We obtain: $p^x - 1 = z^2 - n^{2f+1} - 1$. Now observe that the left-hand side can be factored using the identity:

$$a^x - 1 = (a - 1)(a^{x-1} + a^{x-2} + \cdots + a + 1).$$

Applying this to $p^x - 1$, we get:

$$(p - 1)(p^{x-1} + p^{x-2} + \cdots + p + 1) = z^2 - n^{2f+1} - 1. \tag{3}$$

Let us define the geometric sum as:

$$R = p^{x-1} + p^{x-2} + \cdots + p + 1 = \frac{p^x - 1}{p - 1}.$$

Substituting back into equation (3), we obtain:

$$(p - 1)R = z^2 - n^{2f+1} - 1. \tag{4}$$

To analyze this equation, note that both $p - 1$ and $R$ are positive integers, and their product equals the right-hand side. Since the right-hand side is fixed once $z$ and $n$ are chosen, only specific combinations of $p - 1$ and $R$ can satisfy the equation. We consider two cases:

a) $x = 1$.

Since $R = p^{x-1} + p^{x-2} + \cdots + p + 1 = \frac{p^x - 1}{p-1}$, we have $R = 1$.

However, for any odd prime $p > 3$

$$R = p^{x-1} + p^{x-2} + \cdots + p + 1 > 4 > 1$$

which is a contradiction. Therefore, the case $x = 1$ admits no solution.

b) $x \geq 3$.

Let us assume that the Diophantine equation $p^x + n^y = z^2$, admits a solution in positive integers, where $x$ is an odd integer with $x \geq 3$, and the primes $p$ and $n$ satisfy $p \equiv 2 (\mathrm{mod}\ 3)$ and $n \equiv 1 (\mathrm{mod}\ 3)$, respectively.

Since $p \equiv 2(\mathrm{mod}\ 3)$, and $x$ is odd it follows that $p^x \equiv 2(\mathrm{mod}\ 3)$. Similarly, since $n \equiv 1(\mathrm{mod}\ 3)$, we obtain $n^y \equiv 1(\mathrm{mod}\ 3)$. Therefore, the sum becomes

$$z^2 = p^x + n^y \equiv 2 + 1 \equiv 0(\mathrm{mod}\ 3),$$

which implies that $3|z$. Let us write $z = 3k$ for some $k \in \mathbb{N}$. Substituting into the original equation gives

$$p^x + n^y = z^2 = 9k^2,$$

which yields

$$p^x = 9k^2 - n^y.$$

This expression implies the inequality $p^x < 9k^2$, and hence $p^{\frac{x}{2}} < 3k = z$, which gives the lower bound $z > p^{\frac{x}{2}}$. On the other hand, since $n^y > 0$, we may assume that $n^y < p^x$. we also have $z^2 = p^x + n^y < 2p^x$, then $z < \sqrt{2p^x} = \sqrt{2} \cdot p^{\frac{x}{2}}$. Consequently, the value of $z$ must lie in the open interval $p^{\frac{x}{2}} < z < \sqrt{2} \cdot p^{\frac{x}{2}}$. However, since $x$ is an odd integer, $\frac{x}{2}$ is not an integer, and hence $p^{\frac{x}{2}}$ is irrational. Therefore, the interval $\left(p^{\frac{x}{2}}, \sqrt{2} \cdot p^{\frac{x}{2}}\right)$ contains no integer. This contradicts the assumption that $z$ is a positive integer. In a similar manner, if $n^y \geq p^x$, the same contradiction arises by symmetry of the argument. Thus, under the given assumptions, the equation $p^x + n^y = z^2$ has no solution in positive integers when $x \geq 3$ is odd, $p \equiv 2(\mathrm{mod}\ 3)$, and $n \equiv 1(\mathrm{mod}\ 3)$. Hence, no positive integer solution exists in this case.

From the subcase 4.3.1-4.3.3, we can conclude that the case 4.3 has no solution of non-negative integer. From the proof of 4 cases, it can be concluded that equation (1) are given by $(p, n, x, y, z) \in \{(3, n, 1, 0, 2) \cup (3,\ 2 \cdot 3^s + 1, 2s, 1, 3^s + 1)\}$, for $s$ is a non-negative integer where as $n \equiv 7(\mathrm{mod}\ 12)$ and $\gcd(n, p) = 1$. This completes the proof.

**Corollary 1.** The Diophantine equation $11^x + 19^y = w^4$ has no non-negative integer solution, where $x, y$ and $w$ are non-negative integers.

**Proof.** Suppose that $z = w^2$, so $11^x + 19^y = w^4 = z^2$

From Theorem 2, $11^x + 19^y = w^4$ has no solution in non-negative integers.

**Corollary 2.** The Diophantine equation $p^x + 7^y = h^{2t}$ where $p$ be prime and $p > 3$, has no non-negative integer solution, where $t, x, y$ and $h$ are non-negative integers.

**Proof :** Let $p$ be prime number and $p > 3$

Suppose that $z = h^t$, so $p^x + n^y = h^{2t} = z^2$ .

From Theorem 2 , the Diophantine equation $p^x + 7^y = h^{2t}$ has no solution in non-negative integers.

**Corollary 3.** The Diophantine equation $5^x + n^y = w^4$ where $n \equiv 7(\mathrm{mod}\ 12)$ and $x, y$ are non-negative integers.

**Proof .** Let $p = 5$ and $z = w^2$, so that $5^x + n^y = z^2$ .

From Theorem *2* , the Diophantine equation $5^x + n^y = z^2$ has no solution in non-negative integers.

**Corollary 4.** $7^x + n^y = u^{6k+2}$ where $n \equiv 7 (\mathrm{mod}\ 12)$ for $n, x, y$ and $u$ are non-negative integers.

**Proof.** Let $k$ be a positive integer and $z = u^{3k+1}$. Then the Diophantine equation $7^x + n^y = z^2$. By Theorem 2, this equation has no solutions in non-negative integers.

**Corollary 5.** $17^{2k+1} + n^y = u^{k^2}$ where $n \equiv 7 (\mathrm{mod}\ 12)$ for $n, x, y$ and $u$ are non-negative integers.

**Proof.** Let $k$ be a positive integer, $z = u^k$ and $x = 2k + 1$. Then the Diophantine equation $17^x + n^y = z^2$. By Theorem 2, this equation has no solution in non-negative integers.

## Discussion

The results of this study show that the Diophantine equation $p^x + n^y = z^2$ where $p$ is an odd prime and $n \equiv 7 (\mathrm{mod}\ 12)$ has solutions only in specific cases. When compared with previous works, such as those by Sroysang (2013, 2014), it is evident that Diophantine equation involving prime numbers tend to have few or no solutions. Similarly, Burshtein (2020) demonstrated that the factorization structure involving primes strongly influences the existence of solutions. Dokchan & Pakapongpun (2021) showed that when primes appear in pairs like $p$ and $p + 20$, the equation admits no positive integer solutions. Likewise, Moonchaisook (2022) and Pakapongpun & Chattae (2022) confirmed this trend, illustrating that the structure of equations involving primes and specific constants severely restricts the number of solutions or eliminates them entirely. Furthermore, Tadee (2023, 2024) investigated more complex cubic equations and found that adding certain conditions results in very few or no solutions. This is consistent with the findings of Viriyapong & Viriyapong (2021), who applied modular conditions on $n$ and identified unique specific solutions. Overall, the evidence clearly shows that when Diophantine equations involve primes and modular constraints, the possible solutions become highly restricted. Small modular conditions on $n$ can decisively determine whether the equation has solutions or not. This consistent pattern across multiple studies confirms that analyzing variable constraints is key to solving Diophantine equations. Beyond deepening the understanding of number theory, these findings can support teaching in number theory courses and serve as a foundation for future research into more complex Diophantine equations.

## Conclusions

This research investigated the Diophantine equation $p^x + n^y = z^2$ and proved that it has solutions of the form $(p, n, x, y, z) \in \{(3, n, 1, 0, 2) \cup (3, 2 \cdot 3^s + 1, 2s, 1, 3^s + 1)\}$ where $n, s, x, y$ and $z$ are non-negative integer and $n \equiv 7 (\mathrm{mod}\ 12)$. The proof relies on fundamental concepts in number theory, properties of exponential equations, and the application of Legendre's Theorem, which plays a crucial role in the reasoning. Readers who

are interested may further explore the structure of the proof by considering various values of $n$ and $p$. Furthermore, the results of this research may serve as useful examples in mathematics education, particularly in topics related to number theory and Diophantine equations.

## Acknowledgements

The researcher would like to thank all the experts who have provided assistance and useful suggestions which makes this research article more complete, and accurate. In addition, thank you to the Faculty of Science and Technology , Surindra Rajabhat University for providing good support.

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