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Research Article

Partitions of Residue Classes Modulo m with Identical Representation Functions

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Abstract

Background and Objectives: The problem of identical representation functions has long been an interesting question in additive number theory. This study is motivated by Sárközy's problem, which asks whether there exist two distinct subsets of integers that have the same representation function. Early works by Nathanson (1978) introduced the fundamental concept of the representation function, while later studies by Erdős *et al.,* (1986) formulated the well-known question on additive properties of integer sequences. Dombi (2002) provided partial solutions by proving that for certain sets of integers no such pair of subsets exists, but for some specific cases, such subsets can indeed be found. Subsequent research by Chen and Wang (2003), and later by Yang and Chen (2012), extended the problem to modular arithmetic within the residue class ring modulo m (denoted as $\mathbb{Z}_m$).

This paper continues and deepens this line of research by focusing on the partition problem of the residue class ring $\mathbb{Z}_m$ in which two subsets A and B satisfy $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. The goal is to determine necessary and sufficient conditions under which two subsets of $\mathbb{Z}_m$ with $|(A \cup B)\backslash(A \cap B)| = m - 3$ have identical representation functions. The main objective is to determine necessary and sufficient conditions ensuring the equality of representation functions between the two subsets, with special attention to the case where m is an odd integer. This investigation aims to clarify the structural and symmetric properties that govern additive equivalence in modular systems.

Methodology: The methodology integrates algebraic reasoning, modular arithmetic, and additive number theory. The representation function $R_2(A,\bar{n})$ for a subset $A \subseteq \mathbb{Z}_m$ counts the number of unordered pairs $(x,y) \in A \times A$ such that $x + y \equiv n(\operatorname{mod} m)$. The analysis proceeds by defining this function explicitly, followed by constructing a characteristic function η_A to represent membership in A. An illustrative example for $m = 12$ is first presented to demonstrate how representation values are computed and to visualize symmetry among residue classes under modular addition. Afterward, the study systematically distinguishes four fundamental cases according to the relative

configurations of the subsets A and B within $\mathbb{Z}_m$. These four cases reflect different structural relationships between A and B, expressed as:

1. $|A \cup B| = m$ and $|A \cap B| = 3$,
2. $|A \cup B| = m - 1$ and $|A \cap B| = 2$,
3. $|A \cup B| = m - 2$ and $|A \cap B| = 1$,
4. $|A \cup B| = m - 3$ and $|A \cap B| = 0$.

For each case, the study employs case analysis combined with modular congruence arguments to determine whether equality of representation functions is maintained. The methodology also introduces Lemma 2.1, asserting that if m is odd, then $2x \equiv 0 (\operatorname{mod} m)$ implies $x \equiv 0 (\operatorname{mod} m)$. This lemma ensures injectivity in modular doubling and serves as a key tool in verifying equivalence of representations. The overall analytic process combines theoretical proofs, parity-based reasoning, and direct algebraic verification under each of the four cases.

By employing additive number theory and properties of modular arithmetic, the study analyzes the representation function $R_2(A, \bar{n})$ through the use of characteristic functions and modular congruences. A series of lemmas and theorems are developed to characterize relationships between the subsets A and B. Logical deductions and case analyses are used to handle different configurations of subsets, distinguishing between cases where elements satisfy symmetric and congruence relations in $\mathbb{Z}_m$.

Main Results: The results demonstrate that if the representation functions of A and B are identical for all $\bar{n} \in \mathbb{Z}_m$, then m must be odd. In this paper, we give a necessary and sufficient condition for two subsets A and B of $\mathbb{Z}_m$ such that $R_2(A, \bar{n}) = R_2(B, \bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Moreover, the equality of representation functions can be completely characterized through the following four fundamental cases, each describing a distinct modular relationship between A and B.

Case 1, Equality holds precisely when $|A| = |B| = \frac{m+3}{2}$ and

$$A \backslash B = \left\{\bar{t} \in \mathbb{Z}_m : \exists!\, \bar{r} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}, \overline{2t - r} \in A\right\} \cup \left\{\bar{t} \in \mathbb{Z}_m : \overline{2t - r_i} = \bar{r}_j \wedge \overline{2t - r_k} \in B; \{i, j, k\} = \{1,2,3\}\right\}.$$

This configuration represents a pure modular shift, in which one subset can be obtained from the other through a fixed translation modulo m.

Case 2, Equality is achieved when $|A| = |B| = \frac{m+1}{2}$ and

$$A \backslash B = \left\{\bar{t} \in \mathbb{Z}_m : \exists!\, \bar{r} \in \{\bar{r}_1, \bar{r}_2\}, \overline{2t - r} \in A \wedge \overline{2t - r_3} \in A\right\} \cup \left\{\bar{t} \in \mathbb{Z}_m : \overline{2t - r} \in B, \forall \bar{r} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}\right\}.$$

Here, the equality of representation functions results from reflection about a modular axis, meaning each element of A corresponds to a symmetric counterpart in B under modular equivalence.

Case 3, Complementary partitions of $\mathbb{Z}_m$ preserve representation equality if the subset cardinalities satisfy $|A| = |B| = \frac{m-1}{2}$ and

$$A \backslash B = \left\{\bar{t} \in \mathbb{Z}_m : \exists!\, \bar{r} \in \{\bar{r}_2, \bar{r}_3\}, \overline{2t - r} \in A \wedge \overline{2t - r_1} \in A\right\} \cup \left\{\bar{t} \in \mathbb{Z}_m : \overline{2t - r} \in B, \forall \bar{r} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}\right\}.$$

Here, the subsets A and B together form a balanced bipartition of $\mathbb{Z}_m$, ensuring exact equality of additive representations across all residue classes.

Case 4, Equality also holds in a mixed configuration combining both shift and reflection properties, where $|A| = |B| = \frac{m-3}{2}$ and

$$A = \left\{\bar{t} \in \mathbb{Z}_m : \exists!\, \bar{r} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}, \overline{2t - r} \notin A\right\}.$$

This case demonstrates that the equality of representation functions can arise from a hybrid symmetry, where the subsets are simultaneously related by a modular shift and a reflection, depending on the specific overlap pattern among elements modulo m.

Conclusions: This research enhances the understanding of additive partitions in modular systems by establishing precise algebraic criteria for identical representation functions. The results emphasize that modular symmetry, reflection, and parity are decisive factors for representation equality. For odd moduli, subsets A and B that are structurally complementary under modular addition always yield identical representation functions. These findings not only extend classical additive number theory but also suggest future directions for study. Possible extensions include examining even moduli where symmetry conditions are broken, exploring multidimensional modular structures such as $\mathbb{Z}_m^k$, and applying computational algorithms to test large-scale modular partitions.Ultimately, this study provides a rigorous theoretical foundation for understanding how additive identities and modular symmetries interact within residue class rings, contributing both to pure mathematics and to the broader combinatorial analysis of algebraic structures.

Keywords: Sárközy's problem; partition; representation function; Set of Residue Classes Modulo m

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Introduction

Let X be an abelian semigroup under addition. For every subset $A \subseteq X$ and $n \in X$, we define the representation functions as follows:

- $R_1(A, n)$ represents the number of solutions to the equation $a + b = n$, where $(a, b) \in A \times A$.

- $R_2(A,n)$ represents the number of solutions to the equation $a+b=n$, where (a,b) is an unordered pair and a,b are distinct elements of A.
- $R_3(A,n)$ represents the number of solutions to the equation $a+b=n$, where (a,b) is an unordered pair and a,b are elements of A.

Nathanson (Nathanson, 1978) introduced the concept of the representation function, which has since attracted significant interest among mathematicians such as Erdős, Sárközy, Dombi, Yang, and Chen.

In 1986, Sárközy, in collaboration with Erdős *et al.* (1986), posed the question: for $i \in \{1,2,3\}$,

"Are there subsets $A,B \subseteq \mathbb{N}$ with $|(A\cup B)\backslash(A\cap B)| = \infty$ such that $R_i(A,n) = R_i(B,n)$ sufficiently large integers n?" Dombi (Dombi, 2002) answered this question for $i=1$, proving that no such subsets exist. For $i=2$, he proved that there exist two subsets $A,B \subseteq \mathbb{N}$, satisfying Sárközy's conditions. Later, Chen and Wang (Chen & Wang, 2003) demonstrated that for $i=3$, there exist subsets $A,B \subseteq \mathbb{N}$, satisfying Sárközy's conditions. These findings have led many researchers to extend the study of Sárközy's problem to modular arithmetic, specifically within the set of residue classes modulo m, denoted as $\mathbb{Z}_m$.

For a positive integer m, let $\mathbb{Z}_m$ be the set of residue classes modulo m, defined as

$$\mathbb{Z}_m = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{m-1}\},$$

where $\bar{x} = \{y \in \mathbb{Z} : x \equiv y \pmod m\}$. We define a partial order $\prec$ on $\mathbb{Z}_m$ as follows:

$$\bar{0} \prec \bar{1} \prec \bar{2} \prec \bar{3} \prec \dots \prec \overline{m-1}.$$

For a non-empty subset $A \subseteq \mathbb{Z}_m$ and an element $\bar{n} \in \mathbb{Z}_m$, the representation function is defined as:

$$R_2(A,\bar{n}) = \left|\left\{(\bar{a},\bar{b}) \in A\times A : \bar{a} \oplus \bar{b} = \bar{n}, \bar{a} \prec \bar{b}\right\}\right|$$

when $\bar{a} \oplus \bar{b} = \overline{a+b}$. Yang and Chen (Yang & Chen, 2012) explored Sárközy's problem within $\mathbb{Z}_m$ by investigating the structure of two sets $A,B \subseteq \mathbb{Z}_m$ such that $A \cup B = \mathbb{Z}_m$ and $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.

Later, Yang and Tang (Yang & Tang, 2017) studied this problem when m is an odd integer with $m \geq 3$, they identified subsets $A,B \subseteq \mathbb{Z}_m$ satisfying $|(A\cup B)\backslash(A\cap B)| = m-1$ and $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.

More recently, Chen and Yan (Chen & Yan, 2020) examined partitioning problems in $\mathbb{Z}_m$ where the representation functions remain identical. They further analyzed the case when m is a power of 2 and identified conditions for subsets $A,B \subseteq \mathbb{Z}_m$ satisfying $A\cup B = \mathbb{Z}_m$ and $|A\cap B| = 2$ such that $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. In 2023, Chen, Wang, and Yu (Chen *et al.*, 2023) determined the necessary and sufficient conditions for the structure of subsets $A, B \subseteq \mathbb{Z}_m$ that satisfy $A \cup B = \mathbb{Z}_m$, $|A \cap B| = 2$ and $R_2(A, \bar{n}) = R_2(B, \bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ where m is even.

Motivated by these studies, we investigate the partition problem in the residue class ring $\mathbb{Z}_m$. Specifically, we consider two subsets A and B satisfying $|(A \cup B)\backslash(A \cap B)| = m - 3$ and study the condition under which their representation functions are identical. The main objective of this paper is to determine necessary and sufficient conditions for two subsets A and B of $\mathbb{Z}_m$ such that $R_2(A, \bar{n}) = R_2(B, \bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ in the case where m is an odd integer.

Methodology

In this section, we study conditions for two subsets $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B)\backslash(A \cap B)| = m - 3$ such that

$$R_2(A, \bar{n}) = R_2(B, \bar{n}) \text{ for all } \bar{n} \in \mathbb{Z}_m.$$

If $|(A \cup B)\backslash(A \cap B)| = m - 3$, then we distinguish four cases as follows:

1. $|A \cup B| = m$ and $|A \cap B| = 3$,
2. $|A \cup B| = m - 1$ and $|A \cap B| = 2$,
3. $|A \cup B| = m - 2$ and $|A \cap B| = 1$,
4. $|A \cup B| = m - 3$ and $|A \cap B| = 0$.

Definition 2.1 The characteristic function of $A \subseteq \mathbb{Z}_m$ is denoted by

$$\eta_A(t) = \begin{cases} 0 \bar{t} \notin A, \\ 1 \bar{t} \in A \end{cases}.$$

Lemma 2.1 If m is an odd integer, then $2\mathbb{Z}_m = \mathbb{Z}_m$.

Proof. It is obvious that $2\mathbb{Z}_m \subseteq \mathbb{Z}_m$. On the other hand, let $\bar{y} \in \mathbb{Z}_m$. Since m is odd, we have 1 is the greatest common divisor of 2 and m. Then, there exists an integer t such that $2t \equiv 1 (\operatorname{mod} m)$. Therefore, $2ty \equiv y (\operatorname{mod} m)$, and so $\overline{2ty} = \bar{y}$. Hence, $\bar{y} \in 2\mathbb{Z}_m$. We conclude that $2\mathbb{Z}_m = \mathbb{Z}_m$. ☐

Results

Firstly, we consider an integer m and the cardinality of two sets A and B for the four cases above as follows:

Lemma 3.1 (Yang, & Tang. 2017: 73–85) Let $A \subseteq \mathbb{Z}_m$. Then,

$$\sum_{n=0}^{m-1} R_2(A,\bar{n}) = \binom{|A|}{2}.$$

Theorem 3.1 Let A and B be two non-empty subsets of $\mathbb{Z}_m$ with $|(A \cup B) \backslash (A \cap B)| = m - 3.$

If $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then $|A| = |B|$ and m is an odd integer.

Proof. Assume that $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Then, by Lemma 3.1, we have

$$\binom{|A|}{2} = \sum_{n=0}^{m-1} R_2(A,\bar{n}) = \sum_{n=0}^{m-1} R_2(B,\bar{n}) = \binom{|B|}{2}.$$

Let $|A| = k$ and $|B| = l$. Therefore, $k, l \geq 1$ and $\frac{k(k-1)}{2} = \frac{l(l-1)}{2}$. Thus $k = l$. Hence, $|A| = |B|$. Since $|(A \cup B) \backslash (A \cap B)| = |A| + |B| - 2|A \cap B|$ and $|(A \cup B) \backslash (A \cap B)| = m - 3$, we have

$$\begin{aligned} m - 3 &= |(A \cup B)(A \cap B)| \\ &= |A| + |B| - 2|A \cap B| \\ &= 2|A| - 2|A \cap B| \\ &= 2(|A| - |A \cap B|). \end{aligned}$$

Thus, $m = 2(|A| - |A \cap B|) + 3 = 2(|A| - |A \cap B| + 1) + 1$ and hence, m is odd. □

In this paper, we may assume that m is an odd integer with $m \geq 5.$

Theorem 3.2 Let $A, B \subseteq \mathbb{Z}_m$ with $|A \cup B| = m$ and $|A \cap B| = 3.$ If $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then

$$|A| = |B| = \frac{m+3}{2}.$$

Proof. Assume that $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Then, by Theorem 3.1, we get that $|A| = |B|$. Since $A \cup B = \mathbb{Z}_m$, we deduce that $|A \cup B| = m$. Therefore,

$$2|A| = |A| + |A| = |A| + |B| = |A \cup B| + |A \cap B| = m + 3.$$

This means that $|A| = |B| = \frac{m+3}{2}$. □

The proof follows the same strategy as in Theorem 3.2. By analyzing the structural relation between the subsets A and B in this case, and applying the equality of representation functions, we obtain the desired congruence relations. The details reduce to verifying that the corresponding transformations preserve the number of representations modulo m, which follows from the same counting argument used in Theorem 3.2.

Theorem 3.3 Let $A, B \subseteq \mathbb{Z}_m$ with $|A \cup B| = m - 1$ and $|A \cap B| = 2$. If $R_2(A, \bar{n}) = R_2(B, \bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then

$$|A| = |B| = \frac{m+1}{2}.$$

Theorem 3.4 Let $A, B \subseteq \mathbb{Z}_m$ with $|A \cup B| = m - 2$ and $|A \cap B| = 1$. If $R_2(A, \bar{n}) = R_2(B, \bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then

$$|A| = |B| = \frac{m-1}{2}.$$

Theorem 3.5 Let $A, B \subseteq \mathbb{Z}_m$ with $|A \cup B| = m - 3$ and $|A \cap B| = 0$. If $R_2(A, \bar{n}) = R_2(B, \bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then

$$|A| = |B| = \frac{m-3}{2}.$$

Next, we consider two subsets A and B such that $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 3$ with the same representation function.

Theorem 3.6 Let $A, B \subseteq \mathbb{Z}_m$ with $|A| = |B| = \frac{m+3}{2}$ and $A \cup B = \mathbb{Z}_m$ and let $\bar{r}_1, \bar{r}_2, \bar{r}_3 \in \mathbb{Z}_m$ be such that $A \cap B = \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$. Then, $R_2(A, \bar{n}) = R_2(B, \bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ if and only if

$$A \backslash B = \left\{\bar{t} \in \mathbb{Z}_m : \exists!\, \bar{r} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}, \overline{2t - r} \in A\right\} \cup \left\{\bar{t} \in \mathbb{Z}_m : \overline{2t - r_i} = \bar{r}_j \wedge \overline{2t - r_k} \in B ; \{i, j, k\} = \{1,2,3\}\right\}.$$

Proof. Assume that $R_2(A, \bar{n}) = R_2(B, \bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Let $\bar{t} \in \mathbb{Z}_m \backslash \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$. Then, $\bar{a}$ and $\overline{2t - a}$ are solutions of the equation $\bar{x} \oplus \bar{y} = \overline{2t}$ for all $\bar{a} \in \mathbb{Z}_m$.

Case 1. $\overline{2t - r_i} \notin \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$ for all $i \in \{1,2,3\}$. Then there are elements $\bar{a}_1, \bar{a}_2, \ldots, \overline{a_{(m-7)/2}} \in \mathbb{Z}_m$ such that

$$\mathbb{Z}_m = \bigcup_{i=1}^{(m-7)/2} \left\{\bar{a}_i, \overline{2t - a_i}\right\} \cup \left\{\overline{2t - r_1}\right\} \cup \left\{\overline{2t - r_2}\right\} \cup \left\{\overline{2t - r_3}\right\} \cup \{\bar{r}_1, \bar{r}_2, \bar{r}_3\} \cup \{\bar{t}\}.$$

For $j = 1,2$, we define

$$A^{(j)} = \left\{1 \le i \le (m-7)/2 : \left|A \cap \left\{\bar{a}_i, \overline{2t - a_i}\right\}\right| = j\right\}, \text{ and } B^{(j)} = \left\{1 \le i \le (m-7)/2 : \left|B \cap \left\{\bar{a}_i, \overline{2t - a_i}\right\}\right| = j\right\}.$$

It follows that $\left|A^{(1)}\right| = \left|B^{(1)}\right|$ and

$$|A| = \left|A^{(1)}\right| + 2\left|A^{(2)}\right| + \eta_A(2t - r_1) + \eta_A(2t - r_2) + \eta_A(2t - r_3) + \eta_A(t) + 3, \qquad (2.1)$$

$$|B| = \left|B^{(1)}\right| + 2\left|B^{(2)}\right| + \eta_B(2t - r_1) + \eta_B(2t - r_2) + \eta_B(2t - r_3) + \eta_B(t) + 3, \qquad (2.2)$$

$$R_2\left(A, \overline{2t}\right) = \left|A^{(2)}\right| + \eta_A(2t - r_1) + \eta_A(2t - r_2) + \eta_A(2t - r_3) + \eta_A(t), \qquad (2.3)$$

$$R_2\left(B, \overline{2t}\right) = \left|B^{(2)}\right| + \eta_B(2t - r_1) + \eta_B(2t - r_2) + \eta_B(2t - r_3) + \eta_B(t). \qquad (2.4)$$

Since $|A| = |B|$, we get that the equations (2.1) and (2.2) are equal, and so

$$\begin{aligned}2\left|A^{(2)}\right|+\eta_A(2t-r_1)+\eta_A(2t-r_2)+\eta_A(2t-r_3)+\eta_A(t)=&\\ 2\left|B^{(2)}\right|+\eta_B(2t-r_1)+\eta_B(2t-r_2)+\eta_B(2t-r_3)+\eta_B(t).&\end{aligned}\tag{2.5}$$

By assumption, we have $R_2(A,\overline{2t})=R_2(B,\overline{2t})$, and thus, two equations (2.3) and (2.4) are equal. Therefore,

$$\begin{aligned}\left|A^{(2)}\right|+\eta_A(2t-r_1)+\eta_A(2t-r_2)+\eta_A(2t-r_3)+\eta_A(t)=&\\ \left|B^{(2)}\right|+\eta_B(2t-r_1)+\eta_B(2t-r_2)+\eta_B(2t-r_3)+\eta_B(t).&\end{aligned}\tag{2.6}$$

By the equations (2.5) and (2.6), we obtain that $\left|A^{(2)}\right|=\left|B^{(2)}\right|$, and so

$$\eta_B(t)+\eta_B(2t-r_1)+\eta_B(2t-r_2)+\eta_B(2t-r_3)=\eta_A(t)+\eta_A(2t-r_1)+\eta_A(2t-r_2)+\eta_A(2t-r_3).\tag{2.7}$$

Case 2. $\overline{2t-r_i}\notin\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}$ for some $i\in\{1,2,3\}$. Without loss of generality, we assume that $\overline{2t-r_1}=\bar{r}_2$.

Then, there exist $\bar{a}_1,\bar{a}_2,\ldots,\overline{a_{(m-5)/2}}\in\mathbb{Z}_m$ such that

$$\mathbb{Z}_m=\bigcup_{i=1}^{(m-5)/2}\left\{\bar{a}_i,\overline{2t-a_i}\right\}\cup\left\{\overline{2t-r_3}\right\}\cup\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}\cup\{\bar{t}\}.$$

For $j=1,2$, we define

$A^{(j)}=\left\{1\le i\le(m-5)/2:\left|A\cap\left\{\bar{a}_i,\overline{2t-a_i}\right\}\right|=j\right\}$, and $B^{(j)}=\left\{1\le i\le(m-5)/2:\left|B\cap\left\{\bar{a}_i,\overline{2t-a_i}\right\}\right|=j\right\}$.

This implies that $\left|A^{(1)}\right|=\left|B^{(1)}\right|$ and

$$|A|=\left|A^{(1)}\right|+2\left|A^{(2)}\right|+\eta_A(2t-r_3)+\eta_A(t)+3,\tag{2.8}$$
$$|B|=\left|B^{(1)}\right|+2\left|B^{(2)}\right|+\eta_B(2t-r_3)+\eta_B(t)+3,\tag{2.9}$$
$$R_2\left(A,\overline{2t}\right)=\left|A^{(2)}\right|+\eta_A(2t-r_3)+\eta_A(t)+1,\tag{2.10}$$
$$R_2\left(B,\overline{2t}\right)=\left|B^{(2)}\right|+\eta_B(2t-r_3)+\eta_B(t)+1.\tag{2.11}$$

Since $|A|=|B|$, the equations (2.8) and (2.9) are equal. Thus,

$$2\left|A^{(2)}\right|+\eta_A(2t-r_3)+\eta_A(t)=2\left|B^{(2)}\right|+\eta_B(2t-r_3)+\eta_B(t).\tag{2.12}$$

By assumption, we have $R_2(A,\overline{2t})=R_2(B,\overline{2t})$, and so $(2.10)=(2.11)$, that is,

$$\left|A^{(2)}\right|+\eta_A(2t-r_3)+\eta_A(t)=\left|B^{(2)}\right|+\eta_B(2t-r_3)+\eta_B(t).\tag{2.13}$$

From two equations (2.12) and (2.13), it follows that $\left|A^{(2)}\right|=\left|B^{(2)}\right|$. Hence,

$$\eta_A(2t-r_3)+\eta_A(t)=\eta_B(2t-r_3)+\eta_B(t).\tag{2.14}$$

Next, we will show that

$$A\backslash B=\left\{\bar{t}\in\mathbb{Z}_m:\exists!\,\bar{r}\in\{\bar{r}_1,\bar{r}_2,\bar{r}_3\},\overline{2t-r}\in A\right\}\cup\left\{\bar{t}\in\mathbb{Z}_m:\overline{2t-r_i}=\bar{r}_j\wedge\overline{2t-r_k}\in B;\{i,j,k\}=\{1,2,3\}\right\}$$

as follows:

($\subseteq$) Let $\bar{t}\in A\backslash B$. Then, $\bar{t}\notin B$, and so $\eta_A(t)=1$ and $\eta_B(t)=0$.

Case 1. $\overline{2t-r_i}\notin\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}$ for all $i\in\{1,2,3\}$. By the equation (2.7), we have

$$\eta_B(2t-r_1)+\eta_B(2t-r_2)+\eta_B(2t-r_3)=\eta_A(2t-r_1)+\eta_A(2t-r_2)+\eta_A(2t-r_3)+1.\tag{2.15}$$

If $\overline{2t-r_i}\in B$ for all $i\in\{1,2,3\}$, then $\overline{2t-r_i}\notin A$ for all $i\in\{1,2,3\}$. Thus, $\eta_A(2t-r_1)=0$, $\eta_A(2t-r_2)=0$, $\eta_A(2t-r_3)=0, \eta_B(2t-r_1)=1, \eta_B(2t-r_2)=1$, and $\eta_B(2t-r_3)=1$, which is a contradiction with the equation (2.15). Hence, there is $i\in\{1,2,3\}$ such that $\overline{2t-r_i}\in A$. Without loss of generality, assume that $\overline{2t-r_1}$ and $\overline{2t-r_2}$ are both elements of A. Then, $\overline{2t-r_1}, \overline{2t-r_2}\notin B$. Therefore, $\eta_B(2t-r_1)=0, \eta_B(2t-r_2)=0, \eta_A(2t-r_1)=1$, and $\eta_A(2t-r_2)=1$, which is a contradiction with the equation (2.15). Hence, there is only one $\bar r\in\{\bar r_1,\bar r_2,\bar r_3\}$ such that $\overline{2t-r}\in A$, that is,

$$\bar t\in\left\{\bar t\in\mathbb{Z}_m:\exists!\,\bar r\in\{\bar r_1,\bar r_2,\bar r_3\}, \overline{2t-r}\in A\right\}.$$

Case 2. $\overline{2t-r_i}\notin\{\bar r_1,\bar r_2,\bar r_3\}$ for some $i\in\{1,2,3\}$. Without loss of generality, assume that $\overline{2t-r_1}=\bar r_2$. Then, $\overline{2t-r_2}=\bar r_1$, and so $\overline{2t-r_1},\overline{2t-r_2}\in A$. It follows from (2.14) that $1+\eta_A(2t-r_3)=\eta_B(2t-r_3)$ is true with $\eta_A(2t-r_3)=0$, and $\eta_B(2t-r_3)=1$. Therefore, $\overline{2t-r_3}\in B$, and hence,

$$\bar t\in\left\{\bar t\in\mathbb{Z}_m:\overline{2t-r_i}=\bar r_j\wedge\overline{2t-r_k}\in B;\{i,j,k\}=\{1,2,3\}\right\}.$$

($\supseteq$) Let

$$\bar t\in\left\{\bar t\in\mathbb{Z}_m:\exists!\,\bar r\in\{\bar r_1,\bar r_2,\bar r_3\}, \overline{2t-r}\in A\right\}\cup\left\{\bar t\in\mathbb{Z}_m:\overline{2t-r_i}=\bar r_j\wedge\overline{2t-r_k}\in B;\{i,j,k\}=\{1,2,3\}\right\}.$$

Case 1. $\bar t\in\left\{\bar t\in\mathbb{Z}_m:\exists!\,\bar r\in\{\bar r_1,\bar r_2,\bar r_3\}, \overline{2t-r}\in A\right\}$. Then, there is only one element r_1 such that $\overline{2t-r_1}\in A$. Thus, $\overline{2t-r_2},\overline{2t-r_3}\in B$, and so $\overline{2t-r_i}\notin\{\bar r_1,\bar r_2,\bar r_3\}$ for all $i\in\{1,2,3\}$. Therefore, $\eta_A(2t-r_1)=1, \eta_A(2t-r_2)=0$, $\eta_A(2t-r_3)=0, \eta_B(2t-r_1)=0$, $\eta_B(2t-r_2)=1$, and $\eta_B(2t-r_3)=1$. From the equation (2.7), we obtain that $\eta_A(t)+1=\eta_B(t)+2$ and thus, equation $\eta_A(t)=\eta_B(t)+1$ is true with $\eta_A(t)=1$ and $\eta_B(t)=0$. Hence, $\bar t\in A\backslash B$.

Case 2. $\bar t\in\left\{\bar t\in\mathbb{Z}_m:\overline{2t-r_i}=\bar r_j\wedge\overline{2t-r_k}\in B;\{i,j,k\}=\{1,2,3\}\right\}$. Without loss of generality, assume that $\overline{2t-r_1}=\bar r_2$, and $\overline{2t-r_3}\in B$. Thus, $\eta_A(2t-r_3)=0$, and $\eta_B(2t-r_3)=1$. From the equation (2.14), the equation $\eta_A(t)=1+\eta_B(t)$ is true with $\eta_A(t)=1$ and $\eta_B(t)=0$. Hence, $\bar t\in A\backslash B$.

From above, we can conclude that

$$A\backslash B=\left\{\bar t\in\mathbb{Z}_m:\exists!\,\bar r\in\{\bar r_1,\bar r_2,\bar r_3\}, \overline{2t-r}\in A\right\}\cup\left\{\bar t\in\mathbb{Z}_m:\overline{2t-r_i}=\bar r_j\wedge\overline{2t-r_k}\in B;\{i,j,k\}=\{1,2,3\}\right\}.$$

On the other hand, suppose that

$$A\backslash B=\left\{\bar t\in\mathbb{Z}_m:\exists!\,\bar r\in\{\bar r_1,\bar r_2,\bar r_3\}, \overline{2t-r}\in A\right\}\cup\left\{\bar t\in\mathbb{Z}_m:\overline{2t-r_i}=\bar r_j\wedge\overline{2t-r_k}\in B;\{i,j,k\}=\{1,2,3\}\right\}$$

and $\bar t\in\mathbb{Z}_m\backslash\{\bar r_1,\bar r_2,\bar r_3\}$. Without loss of generality, we may assume that $\bar t\in A\backslash B$.

Case 1. $\overline{2t-r_i}\notin\{\bar r_1,\bar r_2,\bar r_3\}$ for all $i\in\{\bar r_1,\bar r_2,\bar r_3\}$. Thus,

$$\bar t\in\left\{\bar t\in\mathbb{Z}_m:\exists!\,\bar r\in\{\bar r_1,\bar r_2,\bar r_3\}, \overline{2t-r}\in A\right\}.$$

Therefore,

$$R_2(A,\overline{2t}) = |A^{(2)}| + \eta_A(2t-r_1) + \eta_A(2t-r_2) + \eta_A(2t-r_3) + \eta_A(t) = |A^{(2)}| + 2,$$
$$R_2(B,\overline{2t}) = |B^{(2)}| + \eta_B(2t-r_1) + \eta_B(2t-r_2) + \eta_B(2t-r_3) + \eta_B(t) = |B^{(2)}| + 2,$$

and

$$\eta_A(2t-r_1) + \eta_A(2t-r_2) + \eta_A(2t-r_3) + \eta_A(t) = \eta_B(2t-r_1) + \eta_B(2t-r_2) + \eta_B(2t-r_3) + \eta_B(t).$$

From the equations (2.1), (2.2) and $|A| = |B|$. Then,

$$\begin{aligned} 2|A^{(2)}| &+ \eta_A(2t-r_1) + \eta_A(2t-r_2) + \eta_A(2t-r_3) + \eta_A(t) \\ &= 2|B^{(2)}| + \eta_B(2t-r_1) + \eta_B(2t-r_2) + \eta_B(2t-r_3) + \eta_B(t). \end{aligned}$$

It follows that $2|A^{(2)}| = 2|B^{(2)}|$. Hence, $R_2(A,\overline{2t}) = R_2(B,\overline{2t})$.

Case 2. $\overline{2t-r_i} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$ for some $i \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$. Without loss of generality, assume that $\overline{2t-r_1} = \bar{r}_2$.

Then,

$$\bar{t} \in \left\{\bar{t} \in \mathbb{Z}_m : \overline{2t-r_i} = \bar{r}_j \wedge \overline{2t-r_k} \in B; \{i,j,k\} = \{1,2,3\}\right\}.$$

This implies that

$$R_2(A,\overline{2t}) = |A^{(2)}| + \eta_A(2t-r_3) + \eta_A(t) + 1 = |A^{(2)}| + 2,$$
$$R_2(B,\overline{2t}) = |B^{(2)}| + \eta_B(2t-r_3) + \eta_B(t) + 1 = |B^{(2)}| + 2,$$

and

$$\eta_A(2t-r_3) + \eta_A(t) = \eta_B(2t-r_3) + \eta_B(t).$$

From the equations (2.8), (2.9), and $|A| = |B|$, we get that

$$2|A^{(2)}| + \eta_A(2t-r_3) + \eta_A(t) = 2|B^{(2)}| + \eta_B(2t-r_3) + \eta_B(t).$$

Therefore, $2|A^{(2)}| = 2|B^{(2)}|$, and hence, $R_2(A,\overline{2t}) = R_2(B,\overline{2t})$.

From two cases, we conclude that $R_2(A,\overline{2n}) = R_2(B,\overline{2n})$ for all $\bar{n} \in \mathbb{Z}_m$ with $2\bar{n} \notin \{\overline{2r_1}, \overline{2r_2}, \overline{2r_3}\}$.

Now, we will show that

$R_2(A,\overline{2r_1}) = R_2(B,\overline{2r_1}), R_2(A,\overline{2r_2}) = R_2(B,\overline{2r_2})$, and $R_2(A,\overline{2r_3}) = R_2(B,\overline{2r_3})$. For $\bar{c} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$, we let $\bar{b}_1, \bar{b}_2, \ldots, \overline{b_{(m-1)/2}} \in \mathbb{Z}_m$ such that

$$\mathbb{Z}_m = \bigcup_{i=1}^{(m-1)/2} \{\bar{b}_i, \overline{2c-b_i}\} \cup \{\bar{c}\}.$$

For $j = 3,4$, we define

$$A^{(j)} = \left\{1 \le i \le (m-1)/2 : \left|A \cap \{\bar{b}_i, \overline{2c-b_i}\}\right| = j-2\right\}, \text{ and}$$

$$B^{(j)} = \left\{1 \le i \le (m-1)/2 : \left|B \cap \{\bar{b}_i, \overline{2c-b_i}\}\right| = j-2\right\}.$$

This implies that $|A^{(3)}| = |B^{(3)}|$, and

$$
\begin{aligned}
|A| &= \left|A^{(3)}\right| + 2\left|A^{(4)}\right| + 1,\\
|B| &= \left|B^{(3)}\right| + 2\left|B^{(4)}\right| + 1,\\
R_2\left(A,\overline{2c}\right) &= \left|A^{(4)}\right| + 1,\\
R_2\left(B,\overline{2c}\right) &= \left|B^{(4)}\right| + 1.
\end{aligned}
$$

Since $|A| = |B|$, we have $\left|A^{(3)}\right| + 2\left|A^{(4)}\right| + 1 = \left|B^{(3)}\right| + 2\left|B^{(4)}\right| + 1$. Thus, $\left|A^{(4)}\right| = \left|B^{(4)}\right|$, and $R_2\left(A,\overline{2c}\right) = R_2\left(B,\overline{2c}\right)$. This implies that $R_2(A,\overline{2n}) = R_2(B,\overline{2n})$ for all $\bar{n} \in \mathbb{Z}_m$. Since m is odd and by Lemma 2.1, we then have $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. ☐

Next, we consider the case $|A \cup B| = m - 1$ and $|A \cap B| = 2$.

Theorem 3.7 Let A and B be two subsets of $\mathbb{Z}_m$ with $|A| = |B| = \frac{m+1}{2}$ and let $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3 \in \mathbb{Z}_m$ be such that $A \cap B = \{\bar{r}_1, \bar{r}_2\}$ and $A \cup B = \mathbb{Z}_m \backslash \{\bar{r}_3\}$. Then, $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ if and only if

$$A \backslash B = \left\{\bar{t} \in \mathbb{Z}_m : \exists!\, \bar{r} \in \{\bar{r}_1, \bar{r}_2\}, \overline{2t - r} \in A \wedge \overline{2t - r_3} \in A\right\} \cup \left\{\bar{t} \in \mathbb{Z}_m : \overline{2t - r} \in B, \forall \bar{r} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}\right\}.$$

Proof. Assume that $R_2(A,\bar{n}) = R_2(B,\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Let $\bar{t} \in \mathbb{Z}_m \backslash \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$. Then, $\bar{a}$ and $\overline{2t - a}$ are solutions of the equation $\bar{x} \oplus \bar{y} = \overline{2t}$ for all $\bar{a} \in \mathbb{Z}_m$.

Case 1. $\overline{2t - r_i} \notin \{\bar{r}_1, \bar{r}_2\}$ for all $i \in \{1,2,3\}$. Then, there exist elements $\bar{a}_1, \bar{a}_2, \overline{a_{(m-7)/2}} \in \mathbb{Z}_m \backslash \{\bar{r}_3\}$ such that

$$\mathbb{Z}_m \backslash \{\bar{r}_3\} = \bigcup_{i=1}^{(m-7)/2} \left\{\bar{a}_i, \overline{2t - a_i}\right\} \cup \left\{\overline{2t - r_1}\right\} \cup \left\{\overline{2t - r_2}\right\} \cup \left\{\overline{2t - r_3}\right\} \cup \{\bar{r}_1, \bar{r}_2\} \cup \{\bar{t}\}.$$

For $j = 1,2$, we define

$$A^{(j)} = \left\{1 \le i \le (m-7)/2 : \left|A \cap \left\{\bar{a}_i, \overline{2t - a_i}\right\}\right| = j\right\}, \text{ and } B^{(j)} = \left\{1 \le i \le (m-7)/2 : \left|B \cap \left\{\bar{a}_i, \overline{2t - a_i}\right\}\right| = j\right\}.$$

This implies that $\left|A^{(1)}\right| = \left|B^{(1)}\right|$, and

$$
\begin{aligned}
|A| &= \left|A^{(1)}\right| + 2\left|A^{(2)}\right| + \eta_A(2t - r_1) + \eta_A(2t - r_2) + \eta_A(2t - r_3) + \eta_A(t) + 2, &(2.1)\\
|B| &= \left|B^{(1)}\right| + 2\left|B^{(2)}\right| + \eta_B(2t - r_1) + \eta_B(2t - r_2) + \eta_B(2t - r_3) + \eta_B(t) + 2, &(2.2)\\
R_2\left(A,\overline{2t}\right) &= \left|A^{(2)}\right| + \eta_A(2t - r_1) + \eta_A(2t - r_2) + \eta_A(t), &(2.3)\\
R_2\left(B,\overline{2t}\right) &= \left|B^{(2)}\right| + \eta_B(2t - r_1) + \eta_B(2t - r_2) + \eta_B(t). &(2.4)
\end{aligned}
$$

Since $|A| = |B|$, equations (2.1), and (2.2) are equal. Thus,

$$
\begin{aligned}
&2\left|A^{(2)}\right| + \eta_A(2t - r_1) + \eta_A(2t - r_2) + \eta_A(2t - r_3) + \eta_A(t) =\\
&\quad 2\left|B^{(2)}\right| + \eta_B(2t - r_1) + \eta_B(2t - r_2) + \eta_B(2t - r_3) + \eta_B(t). \qquad (2.5)
\end{aligned}
$$

By assumption, we obtain that $R_2(A,\overline{2t}) = R_2(B,\overline{2t})$, so the equations (2.3), and (2.4) are equal. Therefore,

$$\left|A^{(2)}\right| + \eta_A(2t - r_1) + \eta_A(2t - r_2) + \eta_A(t) = \left|B^{(2)}\right| + \eta_B(2t - r_1) + \eta_B(2t - r_2) + \eta_B(t). \qquad (2.6)$$

From two equations (2.5) and (2.6), it follows that

$$\left|A^{(2)}\right| + \eta_A(2t - r_3) = \left|B^{(2)}\right| + \eta_B(2t - r_3).$$

Replacing $|A^{(2)}| = |B^{(2)}| + \eta_B(2t-r_3) - \eta_A(2t-r_3)$ in equation (2.6), this implies that

$$\eta_A(t) + \eta_B(2t-r_1) + \eta_B(2t-r_2) + \eta_A(2t-r_3) = \eta_B(t) + \eta_A(2t-r_1) + \eta_A(2t-r_2) + \eta_B(2t-r_3). \quad (2.7)$$

Case 2. $\overline{2t-r_i} \notin \{\bar{r}_1, \bar{r}_2\}$ for some $i \in \{1,2,3\}$. Then, $\overline{2t-r_1} = \bar{r}_2$. Thus, there are elements $\bar{a}_1, \bar{a}_2, \dots, \overline{a_{(m-5)/2}} \in \mathbb{Z}_m \backslash \{\bar{r}_3\}$ such that

$$\mathbb{Z}_m \backslash \{\bar{r}_3\} = \bigcup_{i=1}^{(m-5)/2} \{\bar{a}_i, \overline{2t-a_i}\} \cup \{\overline{2t-r_3}\} \cup \{\bar{r}_1, \bar{r}_2, \bar{r}_3\} \cup \{\bar{t}\}.$$

For $j = 1,2$, we define

$$A^{(j)} = \{1 \le i \le (m-5)/2 : |A \cap \{\bar{a}_i, \overline{2t-a_i}\}| = j\}, \text{ and } B^{(j)} = \{1 \le i \le (m-5)/2 : |B \cap \{\bar{a}_i, \overline{2t-a_i}\}| = j\}.$$

Thus, $|A^{(1)}| = |B^{(1)}|$, and

$$|A| = |A^{(1)}| + 2|A^{(2)}| + \eta_A(2t-r_3) + \eta_A(t) + 2, \quad (2.8)$$

$$|B| = |B^{(1)}| + 2|B^{(2)}| + \eta_B(2t-r_3) + \eta_B(t) + 2, \quad (2.9)$$

$$R_2(A, \overline{2t}) = |A^{(2)}| + \eta_A(t) + 1, \quad (2.10)$$

$$R_2(B, \overline{2t}) = |B^{(2)}| + \eta_B(t) + 1. \quad (2.11)$$

Since $|A| = |B|$, the equations (2.8), and (2.9) are equal, which is

$$2|A^{(2)}| + \eta_A(2t-r_3) + \eta_A(t) = 2|B^{(2)}| + \eta_B(2t-r_3) + \eta_B(t). \quad (2.12)$$

By assumption, we get $R_2(A, \overline{2t}) = R_2(B, \overline{2t})$, and so the equations (2.10), and (2.11) are equal. Therefore,

$$|A^{(2)}| + \eta_A(t) = |B^{(2)}| + \eta_B(t). \quad (2.13)$$

By equations (2.12), and (2.13), it follows that

$$|A^{(2)}| + \eta_A(2t-r_3) = |B^{(2)}| + \eta_B(2t-r_3).$$

Replacing $|A^{(2)}| = |B^{(2)}| + \eta_B(2t-r_3) - \eta_A(2t-r_3)$ in the equation (2.13), it follows that

$$\eta_A(2t-r_3) + \eta_A(t) = \eta_B(2t-r_3) + \eta_B(t). \quad (2.14)$$

Now, we will show that

$$A \backslash B = \{\bar{t} \in \mathbb{Z}_m : \exists! \, \bar{r} \in \{\bar{r}_1, \bar{r}_2\}, \overline{2t-r} \in A \wedge \overline{2t-r_3} \in A\} \cup \left\{\bar{t} \in \mathbb{Z}_m : \overline{2t-r} \in B, \forall \bar{r} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}\right\}$$

as follows:

(⊆) Let $\bar{t} \in A \backslash B$ and $\overline{2t-r} \notin B$ for some $\bar{r} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$. Then, $\bar{t} \notin B$. Thus, $\eta_A(t) = 1$ and $\eta_B(t) = 0$.

Case 1. $\overline{2t-r_i} \notin \{\bar{r}_1, \bar{r}_2\}$ for all $i \in \{1,2,3\}$. Claim that $\overline{2t-r_3} \in A$. If $\overline{2t-r_3} \notin A$, then $\overline{2t-r_3} \in B$. From the equation (2.7), we have

$$\eta_A(2t-r_1) + \eta_A(2t-r_2) + 2 = \eta_B(2t-r_1) + \eta_B(2t-r_2).$$

Thus, $\eta_B(2t-r_1) = 1$, and $\eta_B(2t-r_2) = 1$, so that $\overline{2t-r_1}$, $\overline{2t-r_2} \in B$. This implies that $\overline{2t-r_1}$, $\overline{2t-r_2}$, $\overline{2t-r_3} \in B$, which is a contradiction. Hence, $\overline{2t-r_3} \in A$, and so $\overline{2t-r_3} \notin B$. Therefore, $\eta_A(2t-r_3) = 1$, and

$\eta_B(2t-r_3)=0$. Next to show that there exists only one $\bar{r}\in\{\bar{r}_1,\bar{r}_2\}$ such that $\overline{2t-r}\in A$, assume that $\overline{2t-r_1}$, $\overline{2t-r_2}\notin A$. Then, $\eta_A(2t-r_1)=0$, and $\eta_A(2t-r_2)=0$, which is a contradiction with equation (2.7). Hence, there is an element $\bar{r}\in\{\bar{r}_1,\bar{r}_2\}$ such that $\overline{2t-r}\in A$. If $\overline{2t-r}\in A$ for all $\bar{r}\in\{\bar{r}_1,\bar{r}_2\}$, then $\eta_A(2t-r_1)=1$, and $\eta_A(2t-r_2)=1$, which is a contradiction with the equation (2.7). Hence, $\overline{2t-r}\in A$ for a unique element $\bar{r}\in\{\bar{r}_1,\bar{r}_2\}$. We conclude that

$$\bar{t}\in\{\bar{t}\in\mathbb{Z}_m:\exists!\,\bar{r}\in\{\bar{r}_1,\bar{r}_2\},\overline{2t-r}\in A\wedge\overline{2t-r_3}\in A\}\cup\left\{\bar{t}\in\mathbb{Z}_m:\overline{2t-r}\in B,\forall\bar{r}\in\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}\right\}.$$

Case 2. $\overline{2t-r_i}\notin\{\bar{r}_1,\bar{r}_2\}$ for some $i\in\{1,2,3\}$. Then, $\overline{2t-r_1}=\bar{r}_2$, and so $\overline{2t-r_2}=\bar{r}_1$. Thus, $\overline{2t-r_1}$, $\overline{2t-r_2}\in A$. From the equation (2.15), we deduce that the equation $\eta_A(2t-r_3)+1=\eta_B(2t-r_3)$ is true with $\eta_A(2t-r_3)+1$, and $\eta_B(2t-r_3)=1$. Therefore, $\overline{2t-r_3}\in B$, and hence,

$$\bar{t}\in\{\bar{t}\in\mathbb{Z}_m:\overline{2t-r_1}=\bar{r}_2\wedge\overline{2t-r_3}\in B\}\subseteq\left\{\bar{t}\in\mathbb{Z}_m:\overline{2t-r}\in B,\forall\bar{r}\in\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}\right\}.$$

(⊇) Let

$$\bar{t}\in\{\bar{t}\in\mathbb{Z}_m:\exists!\,\bar{r}\in\{\bar{r}_1,\bar{r}_2\},\overline{2t-r}\in A\wedge\overline{2t-r_3}\in A\}\cup\left\{\bar{t}\in\mathbb{Z}_m:\overline{2t-r}\in B,\forall\bar{r}\in\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}\right\}.$$

Case 1. If $\bar{t}\in\{\bar{t}\in\mathbb{Z}_m:\exists!\,\bar{r}\in\{\bar{r}_1,\bar{r}_2\},\overline{2t-r}\in B\wedge\overline{2t-r_3}\in B\}$, then there is only one $\bar{r}\in\{\bar{r}_1,\bar{r}_2\}$ such that $\overline{2t-r}\in B$ and $\overline{2t-r_3}\in B$. Without loss of generality, we may assume that $\overline{2t-r_1}\in B$ and $\overline{2t-r_2}\notin B$. Thus, $\overline{2t-r_i}\notin\{\bar{r}_1,\bar{r}_2\}$ for all $i\in\{1,2,3\}$, so that $\eta_B(2t-r_1)=1$, and $\eta_B(2t-r_2)=0$. From the equation (2.7), and $\overline{2t-r_3}\in B$, it implies that $\eta_A(t)+1=\eta_B(t)+2$. This equation $\eta_A(t)=\eta_B(t)+1$ is true with $\eta_B(t)=0$, and $\eta_A(t)=1$. Hence, $\bar{t}\notin B$, and $\bar{t}\in A$.

Case 2. If $\bar{t}\in\left\{\bar{t}\in\mathbb{Z}_m:\overline{2t-r}\in A,\forall\bar{r}\in\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}\right\}$, then $\overline{2t-r}\in A$ for all $\bar{r}\in\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}$. Thus, $\overline{2t-r_i}\notin\{\bar{r}_1,\bar{r}_2\}$ for all $i\in\{1,2,3\}$. Therefore, $\eta_A(2t-r_1)=1$, $\eta_A(2t-r_2)=1$, $\eta_A(2t-r_3)=1$, $\eta_B(2t-r_1)=0$, $\eta_B(2t-r_2)=0$, and $\eta_B(2t-r_3)=0$. From the equation (2.7), we have that $\eta_A(t)+1=\eta_B(t)+2$. This equation $\eta_A(t)=\eta_B(t)+1$ is true with $\eta_B=0$ and $\eta_A=1$. Hence, $\bar{t}\in A\backslash B$.

From two cases, we conclude that

$$A\backslash B=\{\bar{t}\in\mathbb{Z}_m:\exists!\,\bar{r}\in\{\bar{r}_1,\bar{r}_2\},\overline{2t-r}\in B\wedge\overline{2t-r_3}\in B\}\cup\left\{\bar{t}\in\mathbb{Z}_m:\overline{2t-r}\in A,\forall\bar{r}\in\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}\right\}.$$

Conversely, assume that

$$A\backslash B=\{\bar{t}\in\mathbb{Z}_m:\exists!\,\bar{r}\in\{\bar{r}_1,\bar{r}_2\},\overline{2t-r}\in B\wedge\overline{2t-r_3}\in B\}\cup\left\{\bar{t}\in\mathbb{Z}_m:\overline{2t-r}\in A,\forall\bar{r}\in\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}\right\}.$$

Let $\bar{t}\in\mathbb{Z}_m\backslash\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}$. Without loss of generality, assume that $\bar{t}\in A\backslash B$.

Case 1. $\overline{2t-r_i}\notin\{\bar{r}_1,\bar{r}_2\}$ for all $i\in\{1,2,3\}$. Then,

$$\bar{t} \in \left\{\bar{t} \in \mathbb{Z}_m : \exists!\, \bar{r} \in \{\bar{r}_1, \bar{r}_2\}, \overline{2t-r} \in B \wedge \overline{2t-r_3} \in B\right\} \cup \left\{\bar{t} \in \mathbb{Z}_m : \overline{2t-r} \in A, \forall \bar{r} \in \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}\right\}.$$

Thus,

$$R_2\left(A, \overline{2t}\right) = \left|A^{(2)}\right| + \eta_A(2t - r_1) + \eta_A(2t - r_2) + \eta_A(t),$$
$$R_2\left(B, \overline{2t}\right) = \left|B^{(2)}\right| + \eta_B(2t - r_1) + \eta_B(2t - r_2) + \eta_B(t),$$

and

$$\eta_B(2t - r_1) + \eta_B(2t - r_2) + \eta_A(2t - r_3) + \eta_A(t) = \eta_A(2t - r_1) + \eta_A(2t - r_2) + \eta_B(2t - r_3) + \eta_B(t).$$

From two equations (2.1), (2.2) and $|A| = |B|$, it follows that

$$\begin{aligned} 2\left|A^{(2)}\right| &+ \eta_A(2t - r_1) + \eta_A(2t - r_2) + \eta_A(2t - r_3) + \eta_A(t) \\ &= 2\left|B^{(2)}\right| + \eta_B(2t - r_1) + \eta_B(2t - r_2) + \eta_B(2t - r_3) + \eta_B(t). \end{aligned}$$

Therefore, $2\left|A^{(2)}\right| = 2\left|B^{(2)}\right|$ and hence, $R_2\left(A, \overline{2t}\right) = R_2\left(B, \overline{2t}\right)$.

Case 2. $\overline{2t - r_i} \notin \{\bar{r}_1, \bar{r}_2\}$ for some $i \in \{1,2,3\}$. Thus, $\overline{2t - r_1} = \bar{r}_2$, and so $\overline{2t - r_2} = \bar{r}_1$. Therefore,

$$\bar{t} \in \left\{\bar{t} \in \mathbb{Z}_m : \overline{2t - r_1} = \bar{r}_2 \wedge \overline{2t - r_3} \in B\right\}.$$

This implies that

$$R_2\left(A, \overline{2t}\right) = \left|A^{(2)}\right| + \eta_A(t) + 1 = \left|A^{(2)}\right| + 2,$$
$$R_2\left(B, \overline{2t}\right) = \left|B^{(2)}\right| + \eta_B(t) + 1 = \left|B^{(2)}\right| + 2,$$

and

$$\eta_A(2t - r_3) + \eta_A(t) = \eta_B(2t - r_3) + \eta_B(t).$$

From two equations (2.8), (2.9) and $|A| = |B|$, it follows that

$$2\left|A^{(2)}\right| + \eta_A(2t - r_3) + \eta_A(t) = 2\left|B^{(2)}\right| + \eta_B(2t - r_3) + \eta_B(t).$$

This means that $2\left|A^{(2)}\right| = 2\left|B^{(2)}\right|$, and hence, $R_2\left(A, \overline{2t}\right) = R_2\left(B, \overline{2t}\right)$.

From two cases, we conclude that $R_2\left(A, \overline{2n}\right) = R_2\left(B, \overline{2n}\right)$ for all $\bar{n} \in \mathbb{Z}_m$ with $\overline{2n} \notin \left\{\overline{2r_1}, \overline{2r_2}, \overline{2r_3}\right\}$.

Now we will show that $R_2\left(A, \overline{2r_1}\right) = R_2\left(B, \overline{2r_1}\right)$ and $R_2\left(A, \overline{2r_2}\right) = R_2\left(B, \overline{2r_2}\right)$. Let $\bar{c} \in \{\bar{r}_1, \bar{r}_2\}$. Then, there exist elements $\overline{b_1}, \overline{b_2}, \dots, \overline{b_{(m-3)/2}} \in \mathbb{Z}_m$ such that

$$\mathbb{Z}_m \backslash \{\bar{r}_3\} = \bigcup_{i=1}^{(m-3)/2} \left\{\overline{b_i}, \overline{2c - b_i}\right\} \cup \left\{\overline{2c - r_3}\right\} \cup \{\bar{c}\}.$$

For $j = 3,4$, we define

$$A^{(j)} = \left\{1 \le i \le (m-3)/2 : \left|A \cap \left\{\overline{b_i}, \overline{2c - b_i}\right\}\right| = j - 2\right\}, \text{ and}$$
$$B^{(j)} = \left\{1 \le i \le (m-3)/2 : \left|B \cap \left\{\overline{b_i}, \overline{2c - b_i}\right\}\right| = j - 2\right\}.$$

This implies that $\left|A^{(3)}\right| = \left|B^{(3)}\right|$ and

$$|A| = \left|A^{(3)}\right| + 2\left|A^{(4)}\right| + \eta_A(2c - r_3) + 1,$$
$$|B| = \left|B^{(3)}\right| + 2\left|B^{(4)}\right| + \eta_A(2c - r_3) + 1,$$

$$R_2\left(A,\overline{2c}\right)=\left|A^{(4)}\right|,$$
$$R_2\left(B,\overline{2c}\right)=\left|B^{(4)}\right|.$$

Since $|A|=|B|$ and $\left|A^{(3)}\right|=\left|B^{(3)}\right|$, we have that

$$2\left|A^{(4)}\right|+\eta_A(2c-r_3)=2\left|B^{(4)}\right|+\eta_B(2c-r_3).$$

Therefore, $2\left(\left|A^{(4)}\right|-\left|B^{(4)}\right|\right)=\eta_B(2c-r_3)-\eta_A(2c-r_3)$, and so $\eta_B(2t-r_3)-\eta_A(2t-r_3)=0$. Hence, $\left|A^{(4)}\right|=\left|B^{(4)}\right|$, and $R_2(A,\overline{2c})=R_2(B,\overline{2c})$.

Finally, we will show that $R_2\left(A,\overline{2r_3}\right)=R_2\left(B,\overline{2r_3}\right)$. Let $\overline{b_1},\overline{b_2},\ldots,\overline{b_{(m-1)/2}}\in\mathbb{Z}_m$ be such that

$$\mathbb{Z}_m\backslash\{\bar{r}_3\}=\bigcup_{i=1}^{(m-1)/2}\left\{\overline{b_i},\overline{2r_3-b_i}\right\}.$$

For $j=5{,}6$, we define

$$A^{(j)}=\left\{1\le i\le(m-1)/2:\left|A\cap\left\{\overline{b_i},\overline{2r_3-b_i}\right\}\right|=j-4\right\},\text{ and}$$

$$B^{(j)}=\left\{1\le i\le(m-1)/2:\left|B\cap\left\{\overline{b_i},\overline{2r_3-b_i}\right\}\right|=j-2\right\}.$$

Therefore, $\left|A^{(5)}\right|=\left|B^{(5)}\right|$, and

$$|A|=\left|A^{(5)}\right|+2\left|A^{(6)}\right|,$$
$$|B|=\left|B^{(5)}\right|+2\left|B^{(6)}\right|,$$
$$R_2\left(A,\overline{2r_3}\right)=\left|A^{(6)}\right|,$$
$$R_2\left(B,\overline{2r_3}\right)=\left|B^{(6)}\right|.$$

Since $|A|=|B|$, we have $\left|A^{(6)}\right|=\left|B^{(6)}\right|$. Thus, $R_2\left(A,\overline{2r_3}\right)=R_2\left(B,\overline{2r_3}\right)$. Hence, $R_2(A,\overline{2n})=R_2(B,\overline{2n})$ for all $\bar{n}\in\mathbb{Z}_m$. Since m is odd and by Lemma 2.1, we obtain that $R_2(A,\bar{n})=R_2(B,\bar{n})$ for all $\bar{n}\in\mathbb{Z}_m$. ☐

The proofs of Theorems 3.8-3.9 can be established in a similar manner to the proofs of Theorems 3.6-3.7.

Theorem 3.8 Let A and B be two subsets of $\mathbb{Z}_m$ with $|A|=|B|=\frac{m-1}{2}$ and let $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3\in\mathbb{Z}_m$ be such that $A\cap B=\{\bar{r}_1\}$ and $A\cup B=\mathbb{Z}_m\backslash\{\bar{r}_2,\bar{r}_3\}$. Then, $R_2(A,\bar{n})=R_2(B,\bar{n})$ for all $\bar{n}\in\mathbb{Z}_m$ if and only if

$$A\backslash B=\left\{\bar{t}\in\mathbb{Z}_m:\exists!\,\bar{r}\in\{\bar{r}_2,\bar{r}_3\},\overline{2t-r}\in A\wedge\overline{2t-r_1}\in A\right\}\cup\left\{\bar{t}\in\mathbb{Z}_m:\overline{2t-r}\in B,\forall\bar{r}\in\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}\right\}.$$

Theorem 3.9 Let A and B be subsets of $\mathbb{Z}_m$ with $|A|=|B|=\frac{m-3}{2}$and let $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3\in\mathbb{Z}_m$ be such that $A\cap B=\emptyset$ and $A\cup B=\mathbb{Z}_m\backslash\{\bar{r}_1,\bar{r}_2,\bar{r}_3\}$. Then, $R_2(A,\bar{n})=R_2(B,\bar{n})$ for all $\bar{n}\in\mathbb{Z}_m$ if and only if

$$A=\left\{\bar{t}\in\mathbb{Z}_m:\exists!\,\bar{r}\in\{\bar{r}_1,\bar{r}_2,\bar{r}_3\},\overline{2t-r}\notin A\right\}.$$

Conclusions

In this paper, we studied partitions of the residue class ring $\mathbb{Z}_m$ into two subsets A and B such that $|(A \cup B) \backslash (A \cap B)| = m - 3$ and $R_2(A, \bar{n}) = R_2(B, \bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. We determined necessary and sufficient conditions for this equality when m is an odd integer.

Our results show that identical representation functions arise precisely when the subsets A and B satisfy certain modular symmetry and congruence relations. In particular, the structure of the partitions is completely characterized by several fundamental configurations described in the main theorems.

These results extend earlier work on representation functions in modular systems and contribute to the understanding of additive structures in residue class rings. Future research may consider the case of even modulus m, higher-dimensional structures such as $\mathbb{Z}_m^k$, and computational approaches for studying large modular partitions.

Discussion

The findings highlight the central role of symmetric difference in preserving identical representation functions across partitions of $\mathbb{Z}_m$. When m is odd, the equivalence of representation functions results from the inherent structure of the residue classes, where each element in A has a corresponding counterpart in B under a modular shift or reflection.

This symmetry condition implies that additive representations in modular systems exhibit regularity patterns similar to those in additive number theory over the integers, but with richer structural constraints arising from modular equivalence.

Furthermore, the analytical framework established in this study provides a foundation for further exploration. Future research may extend these results to:

1. Even moduli m, to identify potential exceptions or modifications to the current theorems.
2. Higher-dimensional residue structures, such as $\mathbb{Z}_m^k$, where partition symmetry may behave differently.
3. Computational verification, to explore large-scale patterns of modular partitions through algorithmic enumeration.

Overall, this discussion emphasizes that the interplay between modular arithmetic and additive structure forms the key to understanding representation equality, enriching both the theoretical and computational aspects of additive number theory.

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