

Power 3 Mean Labeling of Some Special Graphs

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Abstract

Background and Objectives: Graph labeling, the assignment of integers to the vertices, edges, or both of a graph under specific conditions, has been a central concept in graph theory since its introduction in 1960. Mean labeling, a specific type of graph labeling, was introduced by Somasundaram and Ponraj in 2003 and has since been extensively explored for various graph classes. The concept of Power 3 mean labeling was recently introduced in 2020 by Sreeji and Sandhya. A function f is called a Power 3 mean Labeling of a graph $G = (V, E)$ with p vertices and q edges if it is possible to label the vertices $u \in V$ with distinct labels $f(u)$ from $1, 2, \dots, q+1$ in such a way

that when $f(u) = x$ and $f(v) = y$, each edge $e = uv$ is assigned a label defined by $f(e) = \left\lceil \left(\frac{x^3 + y^3}{2} \right)^{\frac{1}{3}} \right\rceil$ or

$\left\lfloor \left(\frac{x^3 + y^3}{2} \right)^{\frac{1}{3}} \right\rfloor$. In this case, f is a Power 3 mean labeling of G and G is called a Power 3 mean graph. Sreeji and

Sandhya demonstrated that some graphs possess this property while others do not. Their subsequent work explored this labeling for the line graphs of certain graphs. Moreover, they considered Power 3 mean labeling for various graphs resulting from the duplication of graph elements. This paper contributes to this area of study by proving that Ladder graphs, Total graphs of a path and Chains of cycles are a family of Power 3 mean graphs. **n -ladder graph** can be defined as the Cartesian product of two paths, $L_n = P_2 \times P_n$, and it is equivalent to the $2 \times n$ grid graph. **Total graph of G** , denoted by $T(G)$, is the graph whose vertex set consists of all vertices and edges of G . Two vertices in $T(G)$ are adjacent if and only if their corresponding elements in G are either adjacent or incident in G . **Chain of m cycle of length n** is a graph obtained by taking m copies of the cycle graph C_n ($n \geq 3, m \geq 1$), arranging them in linear sequence, and identifying exactly one vertex of the i -th copy with one vertex of the $(i+1)$ -th copy for $i = 1, \dots, m-1$. We denote this chain by C_n^m .

Methodology: To study Power 3 mean graphs, we begin by considering the relationships between the numbers assigned to an edge uv when the endpoints are $f(u) = i$ and $f(v) = i+m$. For $m=1, \dots, 5$, the lower and upper bounds of the possible 3 mean values can be determined, and a labeling pattern for the vertices and edges that satisfies the given conditions can be identified. Next, for $n \geq 1$, we prove by induction on n to show that L_n and $T(P_n)$ are Power 3 mean graphs. Finally, for $n \geq 3$ and $m \geq 1$ the Power 3 mean labeling of C_n^m is considered in two cases: when n is even and when n is odd.

Main Results: We establish four lemmas describing the possible integer values assigned to an edge uv when $f(u)=i$ and $f(v) = i+m$, the results are summarized as follows:

- | | |
|--------------------|--|
| (1) $m = 1$ | , $f(uv) = i$ or $i+1$ |
| (2) $m = 2$ or 3 | , $f(uv) = i+1$ or $i+2$ |
| (3) $m = 4$ | , $f(uv) = i+2$ or $i+3$ |
| (4) $m = 5$ | , $f(uv) = i+3$ or $i+4$ when $3 \leq i \leq 9$ and
$f(uv) = i+2$ or $i+3$ when $i \geq 10$. |

We obtain three theorems presenting the Power 3 mean labeling functions of the ladder graph, the total graph of a path, and the chain of m cycles of length n .

Conclusions: The Ladder graph, Total graph of path and Chain of m cycle of length n are Power 3 mean graphs. To determine whether a class of graphs admits a Power 3 mean labeling, one common approach is to identify a suitable labeling pattern for the vertices and edges that satisfies the given conditions. This is typically achieved by analyzing the relationships among the integers assigned to the end vertices and the resulting values obtained for the corresponding edges. In the case of Power 3 mean labeling, when there is a large difference between the numbers of edges and vertices, the structure of the graph becomes more complex. As a result, it becomes difficult to determine an appropriate labeling pattern for the vertices. This difficulty also arises in graphs that form a closed loop, where finding a consistent labeling pattern under the given conditions is particularly challenging. The study of graph labeling remains a topic of significant interest among mathematicians, with research conducted under various conditions and on different types of graphs. For Power 3 mean graphs, there remain several classes of graphs that are of interest to investigate in order to determine whether they admit a Power 3 mean labeling.

Keywords: Power 3 mean graph; Ladder graph; Total graph; Chain of cycle graph

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Introduction

Graph labeling, the assignment of integers to the vertices, edges, or both of a graph under specific conditions, has been a central concept in graph theory since its introduction in 1960. For a detailed survey of graph labeling, we refer to Gallian (Gallian, 2022). Mean labeling, a specific type of graph labeling, was introduced by Somasundaram and Ponraj in 2003 (Somasundaram & Ponraj, 2003) and has since been extensively explored for various graph classes. The concept of Power 3 mean labeling was recently introduced in 2020 by Sreeji and Sandhya (Sreeji & Sandhya, 2020). They demonstrated that some graphs possess this property while others do not. Their subsequent work explored this labeling for the line graphs of certain graphs. Moreover, they considered Power 3 mean labeling for various graphs resulting from the duplication of graph elements.

This paper contributes to this area of study by proving that Ladder graphs, Total graphs of a path and chains of m cycles of length n form a family of Power 3 mean graphs. Standard graph terminology and notation follow West (West, 2021).

Definition 1

A graph $G = (V, E)$ with p vertices and q edges is a Power 3 mean graph if there exists a labeling function $f: V \rightarrow \{1, 2, \dots, q+1\}$ such that the vertex labels are distinct, and the induced edge labeling $f(e)$

for any edge $e = uv$ is defined as: $f(e) = \left\lfloor \left(\frac{x^3 + y^3}{2} \right)^{\frac{1}{3}} \right\rfloor$ or $\left\lceil \left(\frac{x^3 + y^3}{2} \right)^{\frac{1}{3}} \right\rceil$ where $f(u) = x$ and $f(v) = y$

are the vertex labels. The function f is called a Power 3 mean labeling.

Example 2 Figure 1 shows that the graph P_6 is a Power 3 mean graph.

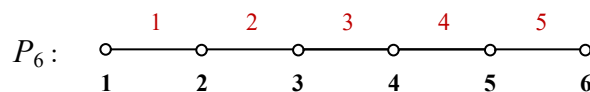


Figure 1 The labeling of P_6

Remark 3 If G is a Power 3 mean graph then the vertices are labeled from $1, 2, \dots, q+1$ and the edges are labels from $1, 2, \dots, q$.

Remark 4 If G is a Power 3 mean graph then the vertices must include labels '1' and '2', and these vertices must be adjacent to ensure that an edge receives the label '1'.

Definition 5

n -ladder graph can be defined as the Cartesian product of two paths, $L_n = P_2 \times P_n$, and it is equivalent to the $2 \times n$ grid graph.

Definition 6

Total graph of G , denoted by $T(G)$, is the graph whose vertex set consists of all vertices and edges of G . Two vertices in $T(G)$ are adjacent if and only if their corresponding elements in G are either adjacent or incident in G .

Definition 7

Chain of m cycle of length n is a graph obtained by taking m copies of the cycle graph C_n ($n \geq 3$, $m \geq 1$), arranging them in linear sequence, and identifying exactly one vertex of the i -th copy with one vertex of the $(i+1)$ -th copy for $i = 1, \dots, m-1$. We denote this chain by C_n^m .

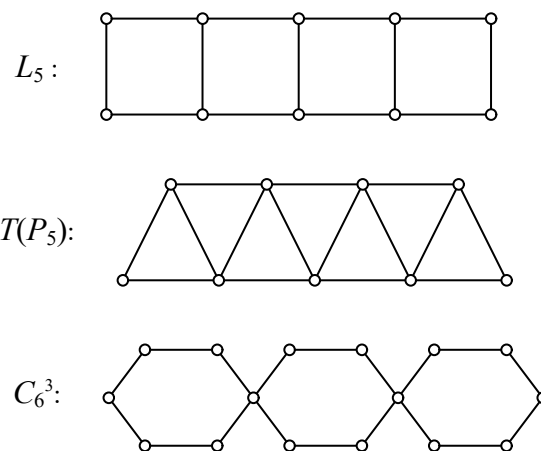


Figure 2 The graphs L_5 , $T(P_5)$ and C_6^3

Methodology

For a Power 3 mean graph, we begin by considering the relationships between the numbers assigned to an edge uv when the endpoints are $f(u) = i$ and $f(v) = i + m$. For $m = 1, \dots, 5$, the lower and upper bounds of the possible 3 mean values can be determined, and a labeling pattern for the vertices and edges that satisfies the given conditions can be identified. Next, for $n \geq 1$, we prove by induction on n to show that L_n and $T(P_n)$ are Power mean graphs. Finally, for $n \geq 3$ and $m \geq 1$ the Power 3 mean labeling of C_n^m is considered in two cases: when n is even and when n is odd.

Results

Let G be a Power 3 mean graph and let uv is an edge in G . We labeled end points of uv with integers i and $i + m$. It is easy to see that when $m = 1$, $f(uv) = i$ or $i + 1$. Next, we consider the relations between the numbers assigned to the edge uv when $m = 2, 3, 4, 5$, as shown in the following lemmas.

Lemma 8 If u and v are the vertices with label i and $i + 2$ of a Power 3 mean graph then $f(uv) = i + 1$ or $i + 2$.

$$\begin{aligned} \text{Proof: } f(uv) &= \left[\frac{i^3 + (i+2)^3}{2} \right]^{\frac{1}{3}} \\ &= \left[\frac{i^3 + (i^3 + 3i^2(2) + 3i(4) + 8)}{2} \right]^{\frac{1}{3}} \\ &= \left[i^3 + 3i^2 + 3i(2) + 4 \right]^{\frac{1}{3}} \\ &> \left[i^3 + 3i^2 + 3i + 1 \right]^{\frac{1}{3}} \\ &= \left[(i+1)^3 \right]^{\frac{1}{3}} \\ &= i + 1 \end{aligned}$$

Since $f(uv)$ is an integer and $f(uv) > i + 1$, $f(uv) = i + 1$ or $i + 2$.

Lemma 9 If u and v are the vertices with label i and $i + 3$ of a Power 3 mean graph. For $i \geq 3$ then $f(uv) = i + 1$ or $i + 2$.

$$\begin{aligned}
 \text{Proof: } f(uv) &= \left[\frac{i^3 + (i+3)^3}{2} \right]^{\frac{1}{3}} \\
 &= \left[\frac{i^3 + (i^3 + 3i^2(3) + 3i(9) + 27)}{2} \right]^{\frac{1}{3}} \\
 &= \left[i^3 + 3i^2(1.5) + 3i(4.5) + 13.5 \right]^{\frac{1}{3}} \\
 &> \left[i^3 + 3i^2 + 3i + 1 \right]^{\frac{1}{3}} \\
 &= \left[(i+1)^3 \right]^{\frac{1}{3}} \\
 &= i+1
 \end{aligned}$$

$$\begin{aligned}
 \text{and } f(uv) &= \left[i^3 + 3i^2(1.5) + 3i(4.5) + 13.5 \right]^{\frac{1}{3}} \\
 &< \left[i^3 + 3i^2(1.5) + 3i^2(0.5) + 3i(4.5) - 3i(0.5) + 13.5 - 5.5 \right]^{\frac{1}{3}} \\
 &= \left[i^3 + 3i^2(2) + 3i(4) + 8 \right]^{\frac{1}{3}} \\
 &= \left[(i+2)^3 \right]^{\frac{1}{3}} \\
 &= i+2
 \end{aligned}$$

Since $f(uv)$ is an integer, $f(uv) = i+1$ or $i+2$.

Lemma 10 If u and v are the vertices with label i and $i+4$ of a Power 3 mean graph. For $i \geq 3$ then $f(uv) = i+2$ or $i+3$.

$$\begin{aligned}
 \text{Proof: } f(uv) &= \left[\frac{i^3 + (i+4)^3}{2} \right]^{\frac{1}{3}} \\
 &= \left[i^3 + 3i^2(2) + 3i(8) + 32 \right]^{\frac{1}{3}} \\
 &> \left[i^3 + 3i^2(2) + 3i(4) + 8 \right]^{\frac{1}{3}} \\
 &= \left[(i+2)^3 \right]^{\frac{1}{3}} \\
 &= i+2
 \end{aligned}$$

$$\begin{aligned} \text{and } f(uv) &= \left[i^3 + 3i^2(2) + 3i(8) + 32 \right]^{\frac{1}{3}} \\ &< \left[i^3 + 3i^2(3) + 3i(9) + 27 \right]^{\frac{1}{3}} \\ &= \left[(i+3)^3 \right]^{\frac{1}{3}} \\ &= i+3 \end{aligned}$$

Since $f(uv)$ is an integer, $f(uv) = i+2$ or $i+3$.

Lemma 11 If u and v are the vertices with label i and $i+5$ of a Power 3 mean graph, then

$$f(uv) = \begin{cases} i+3 & \text{or } i+4, & 3 \leq i \leq 9 \\ i+2 & \text{or } i+3, & i \geq 10 \end{cases}$$

Proof:

Case 1 for $3 \leq i \leq 9$,

$$\begin{aligned} f(uv) &= \left[\frac{i^3 + (i+5)^3}{2} \right]^{\frac{1}{3}} \\ &= \left[i^3 + 3i^2(2.5) + 3i(12.5) + 62.5 \right]^{\frac{1}{3}} \\ &> \left[i^3 + 3i^2(2.5) + 3i^2(0.5) + 3i(12.5) - 3i(3.5) + 62.5 - 35.5 \right]^{\frac{1}{3}}, \\ &= \left[i^3 + 3i^2(3) + 3i(9) + 27 \right]^{\frac{1}{3}} \\ &= \left[(i+3)^3 \right]^{\frac{1}{3}} \\ &= i+3 \end{aligned}$$

$$\begin{aligned} \text{and } f(uv) &= \left[i^3 + 3i^2(2.5) + 3i(12.5) + 62.5 \right]^{\frac{1}{3}} \\ &< \left[i^3 + 3i^2(4) + 3i(16) + 64 \right]^{\frac{1}{3}} \\ &= \left[(i+4)^3 \right]^{\frac{1}{3}} \\ &= i+4 \end{aligned}$$

Since $f(uv)$ is an integer, $f(uv) = i+3$ or $i+4$.

Case 2 for $i \geq 10$,

$$\begin{aligned} f(uv) &= \left[\frac{i^3 + (i+5)^3}{2} \right]^{\frac{1}{3}} \\ &= \left[i^3 + 3i^2(2.5) + 3i(12.5) + 62.5 \right]^{\frac{1}{3}} \\ &> \left[i^3 + 3i^2(2) + 3i(4) + 8 \right]^{\frac{1}{3}} \\ &= i + 2 \end{aligned}$$

$$\begin{aligned} \text{and } f(uv) &= \left[i^3 + 3i^2(2.5) + 3i(12.5) + 62.5 \right]^{\frac{1}{3}} \\ &< \left[i^3 + 3i^2(2.5) + 3i^2(0.5) + 3i(12.5) - 3i(3.5) + 64 - 37 \right]^{\frac{1}{3}} \\ &= \left[i^3 + 3i^2(3) + 3i(9) + 27 \right]^{\frac{1}{3}} \\ &= i + 3 \end{aligned}$$

Since $f(uv)$ is an integer, $f(uv) = i + 2$ or $i + 3$.

Next, we consider the labeling of n -ladder graph. Graph $L_n = P_2 \times P_n$ has $2n$ vertices and $3n - 2$ edges. Let vertex set of L_n be $V(L_n) = \{u_1, \dots, u_n\} \cup \{v_1, \dots, v_n\}$ and edge set be $E(L_n) = \{e_i = u_i u_{i+1}\} \cup \{k_i = v_i v_{i+1}\} \cup \{l_j = u_j v_j\}$ for $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, n$.

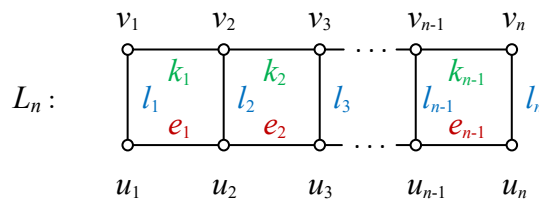


Figure 3 The graph L_n

Theorem 12 For $n \geq 1$, L_n is a Power 3 mean graph.

Proof: It is easy to see that $L_1 \cong P_2$ and $L_2 \cong C_4$, then they are Power 3 mean graph. Next, we consider when $n \geq 3$. We prove by induction on n .

For $n = 3$, L_3 is a Power 3 mean graph.

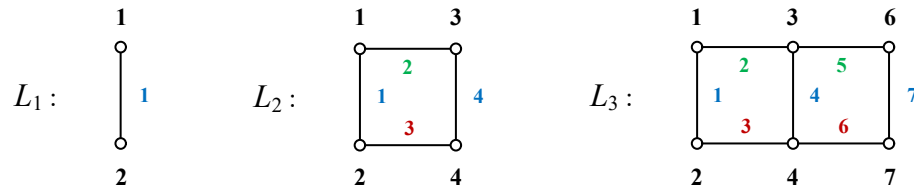


Figure 4 The labeling of L_1 , L_2 and L_3

Let $n \geq 4$. We assume that L_{n-1} is a Power 3 mean graph, with $3n-5$ edges. Without loss of generality, we may assume that $f(v_{n-1}) = 3n-6$ and $f(u_{n-1}) = 3n-5$. For L_n , we add two vertices v_n and u_n with $f(v_n) = 3n-3$, $f(u_n) = 3n-2$. And labeling new three edges with $f(k_{n-1}) = f(v_{n-1}v_n) = 3n-4$, $f(e_{n-1}) = f(u_{n-1}u_n) = 3n-3$ and $f(l_n) = f(v_nu_n) = 3n-2$. By lemmas 8 - 11 then the resulting graph is a Power 3 mean graph. This completes the proof.

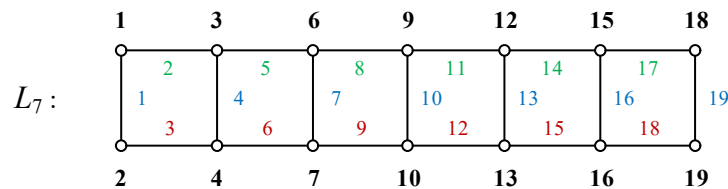


Figure 5 The labeling of L_7

Next, we consider the labeling of the total graph $T(P_n)$. For $n \geq 2$, $T(P_n)$ has $2n-1$ vertices and $4n-5$ edges. Let vertex set of $T(P_n)$ be $V(T(P_n)) = \{u_1, \dots, u_n\} \cup \{v_1, \dots, v_{n-1}\}$ and edge set be $E(T(P_n)) = \{e_i = u_iu_{i+1}\} \cup \{l_i = u_iv_i\} \cup \{s_i = v_iu_{i+1}\} \cup \{k_j = v_jv_{j+1}\}$ for $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, n-2$.

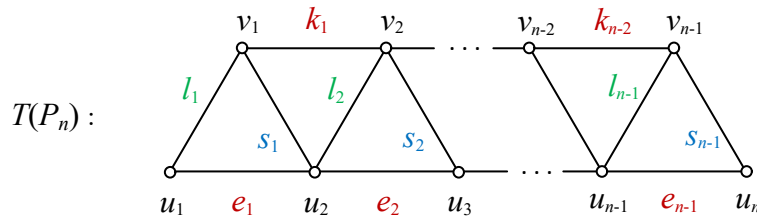


Figure 6 The graph $T(P_n)$

Theorem 13 For $n \geq 1$, $T(P_n)$ is a Power 3 mean graph.

Proof : It is easy to see that $T(P_1) \cong P_1$ and $T(P_2) \cong C_3$, then they are a Power 3 mean graph. Next, we consider when $n \geq 3$. We prove by induction on n . For $n = 3$, $T(P_3)$ is a Power 3 mean graph.

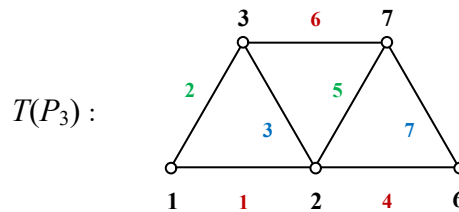


Figure 7 The labeling of $T(P_3)$

Let $n \geq 4$. We assume that $T(P_{n-1})$ is a Power 3 mean graph, with $4n-9$ edges. Without loss of generality, we may assume that $f(v_{n-2}) = 4n-9$ and $f(u_{n-1}) = 4n-10$. For $T(P_n)$, we add two vertices v_{n-1} and u_n with $f(v_{n-1}) = 4n-5$, $f(u_n) = 4n-6$. And labeling new four edges with $f(e_{n-1}) = f(u_{n-1}u_n) = 4n-8$, $f(l_{n-1}) = f(u_{n-1}v_{n-1}) = 4n-7$, $f(k_{n-2}) = f(v_{n-1}v_{n-2}) = 4n-6$ and $f(s_{n-1}) = f(v_{n-1}u_n) = 4n-5$. By lemmas 8 - 11 then the resulting graph is a Power 3 mean graph. This completes the proof.

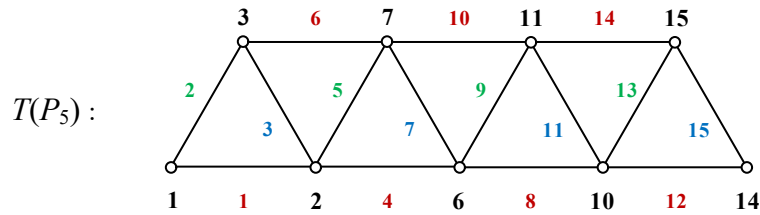


Figure 8 The labeling of $T(P_5)$

Finally, we consider the labeling of the chain of m cycle of length n . For $n \geq 3$ and $m \geq 1$, C_n^m has $m(n-1)+1$ vertices and mn edges. We consider chain of m cycle of length n in two cases, when n is even and when n is odd.

Theorem 14 For $n \geq 3$ and $m \geq 1$, C_n^m is a Power 3 mean graph.

Proof

Case 1 when n is even. For $i = 1, 2, \dots, m$, $V(C_n^m) = \{u_{i1}, u_{i2}, \dots, u_{in}\}$. Let the vertex $u_{i(\frac{n}{2}+1)}$ of C_n^i is a

common vertex with $u_{(i+1)1}$ of C_n^{i+1} . For $m \geq 1$ and $n \geq 3$, we define a function

$f: V(C_n^m) \rightarrow \{1, 2, \dots, mn+1\}$ by

$$f(u_{mp}) = \begin{cases} 1 & ; \quad p=1 \\ n(m-1)+2(p-1) & ; \quad 2 \leq p \leq \frac{n}{2}+1 \\ n(m-1)+2(n-p)+3 & ; \quad \frac{n}{2}+1 < p \leq n \end{cases}$$

For $i = 1, 2, \dots, m$, $E(C_n^m) = \{e_{i1}, e_{i2}, \dots, e_{in}\}$ where $e_{mi} = u_{mi}u_{m(i+1)}$. The edges are labeled with

$$f(u_{mp}u_{m(p+1)}) = \begin{cases} n(m-1)+2p-1 & ; \quad 1 \leq p \leq \frac{n}{2} \\ n(m-1)+2(n-p+1) & ; \quad \frac{n}{2} < p \leq n \end{cases}$$

Obviously, f is a Power 3 mean labeling. Hence C_n^m is a Power 3 mean graph.

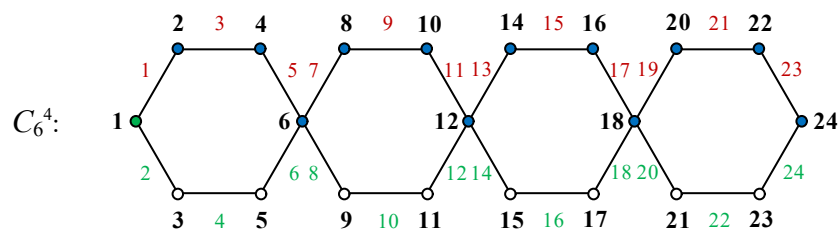


Figure 9 The labeling of C_6^4

Case 2 when n is odd. Let $V(C_n^m) = \{u_{i1}, u_{i2}, \dots, u_{in}\}$ for $i = 1, 2, \dots, m$. The vertex $u_{i(\frac{n+3}{2})}$ of C_n^i is a

common vertex with $u_{(i+1)1}$ of C_n^{i+1} . For $m \geq 1$ and $n \geq 3$, define a function

$f: V(C_n^m) \rightarrow \{1, 2, \dots, mn+1\}$ by

$$f(u_{mp}) = \begin{cases} 1 & ; \quad p=1 \\ n(m-1)+2(p-1) & ; \quad 2 \leq p \leq \frac{n+1}{2} \\ n(m-1)+2(n-p)+3 & ; \quad \frac{n+1}{2} < p \leq n \end{cases}$$

For $i = 1, 2, \dots, m$, $E(C_n^m) = \{e_{i1}, e_{i2}, \dots, e_{in}\}$ where $e_{mi} = u_{mi}u_{m(i+1)}$. The edges are labeled with,

$$f(u_{mp}u_{m(p+1)}) = \begin{cases} n(m-1)+2p-1 & ; 1 \leq p \leq \frac{n+1}{2} \\ n(m-1)+2(n-p+1) & ; \frac{n+1}{2} < p \leq n \end{cases}$$

Obviously, f is a Power 3 mean labeling. Hence C_n^m is a Power 3 mean graph.

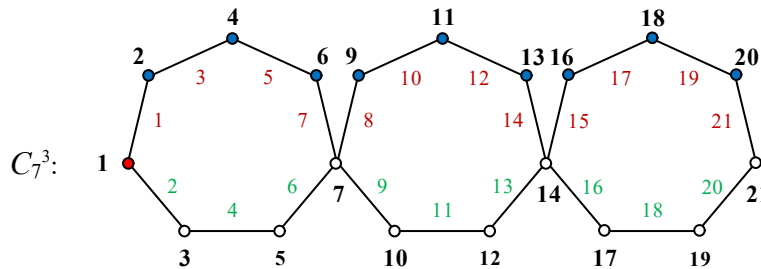


Figure 10 The labeling of C_7^3

Discussion

To determine whether a class of graphs admits a Power 3 mean labeling, one common approach is to identify a suitable labeling pattern for the vertices and edges that satisfies the given conditions. This is typically achieved by analyzing the relationships among the integers assigned to the end vertices and the resulting values obtained for the corresponding edges. In the case of Power 3 mean labeling, when there is a large difference between the numbers of edges and vertices, the structure of the graph becomes more complex. As a result, it becomes difficult to determine an appropriate labeling pattern for the vertices. This difficulty also arises in graphs that form a closed loop, where finding a consistent labeling pattern under the given conditions is particularly challenging. The study of graph labeling remains a topic of significant interest among mathematicians, with research conducted under various conditions and on different types of graphs. For Power 3 mean graphs, there remain several classes of graphs that are of interest to investigate in order to determine whether they admit a Power 3 mean labeling.

Conclusions

The Ladder graph, Total graph of path and Chain of m cycle of length n are Power 3 mean graph.

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