

A Hybrid Safeguarded Quasi-Newton Method for Nonlinear Systems with Engineering Applications

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Abstract

Background and Objectives: Nonlinear systems of equations arise extensively in engineering, applied physics, computational science, economics, and industrial process modeling. Accurate and efficient numerical techniques for solving such systems are therefore essential in many scientific and engineering applications. Classical Newton's method remains one of the most widely used approaches because of its rapid local quadratic convergence. However, its implementation requires repeated evaluations and factorizations of the Jacobian matrix at every iteration, which can become computationally expensive for large-scale or moderately complex nonlinear systems. In addition, Newton's method is highly sensitive to the choice of initial approximations and may fail to converge when the Jacobian matrix is singular, ill-conditioned, or poorly approximated near the solution. To reduce computational cost, quasi-Newton methods such as Broyden's method replace exact Jacobian evaluations with low-rank matrix updates. These methods significantly improve efficiency by avoiding repeated Jacobian computations, making them attractive for practical applications involving expensive function evaluations. Nevertheless, classical Broyden-type methods may experience instability, stagnation, inaccurate search directions, and error accumulation due to progressive deterioration of the inverse Jacobian approximation during the iterative process. Such difficulties often reduce convergence reliability, particularly for strongly nonlinear or ill-conditioned problems. Motivated by these limitations, this study proposes a Hybrid Safeguarded Quasi-Newton Method (HSQNM) that combines the efficiency of quasi-Newton updates with additional stabilization and safeguarding mechanisms to improve robustness, reliability, and overall convergence performance.

Methodology: The proposed HSQNM integrates an inverse Broyden rank-one update with three complementary stabilization components: adaptive damping, curvature-based safeguarding, and restart-based recalibration. The adaptive damping mechanism dynamically adjusts the step length according to the behavior of the nonlinear residual, ensuring sufficient residual reduction and preventing excessively large iterative steps that may lead to divergence. A curvature-monitoring criterion is incorporated to detect unstable or unreliable updates when the secant condition becomes nearly degenerate or when the approximate inverse Jacobian loses numerical quality.

Under such conditions, a safeguarding strategy modifies the update process to maintain stability and improve descent behavior. Furthermore, a restart mechanism is introduced to periodically reconstruct the inverse Jacobian approximation whenever deterioration or stagnation is detected. This recalibration process reduces accumulated numerical errors and restores the quality of the iterative search direction. Theoretical analysis of the proposed method is developed under standard assumptions of Lipschitz continuity and nonsingularity of the Jacobian matrix in a neighborhood of the exact solution. Local convergence properties are established to demonstrate that the proposed hybrid strategy preserves the desirable convergence characteristics of quasi-Newton methods while improving robustness in challenging nonlinear regimes.

Main Results: To evaluate the effectiveness of the proposed algorithm, numerical experiments were conducted on a collection of benchmark nonlinear systems together with several engineering applications involving nonlinear mathematical models. The benchmark problems include systems with varying dimensions, nonlinearities, and conditioning characteristics to examine both efficiency and stability. Performance comparisons were carried out against the classical Broyden method and Newton's method using metrics such as iteration count, residual norm reduction, Jacobian evaluations, computational cost, and convergence success rate. The numerical results demonstrate that HSQNM consistently achieves lower failure rates and improved convergence stability compared with the classical Broyden method. In many test problems, the proposed approach required fewer Jacobian recalculations while maintaining accurate convergence behavior. The adaptive damping strategy was particularly effective in preventing divergence for poor initial guesses, whereas the restart mechanism successfully mitigated stagnation caused by inaccurate inverse Jacobian approximations. Additional experiments on engineering nonlinear systems, including diode circuit analysis and radiative heat transfer models, further confirmed the practical applicability of the proposed method. In these applications, HSQNM achieved convergence accuracy comparable to Newton's method but with reduced computational expense and improved robustness. The results also indicate that the safeguarding framework enhances reliability for ill-conditioned systems where traditional quasi-Newton methods may fail or exhibit unstable iterative behavior.

Conclusions: The proposed Hybrid Safeguarded Quasi-Newton Method provides a reliable and computationally efficient framework for solving moderate-scale nonlinear systems arising in engineering and scientific applications. By integrating adaptive damping, curvature-based safeguarding, and restart-based recalibration into the inverse Broyden update process, the method improves convergence robustness without sacrificing the computational advantages of quasi-Newton techniques. Numerical experiments demonstrate that the proposed approach offers better stability, lower failure rates, and competitive computational performance compared with classical Broyden-

type methods. Moreover, the method retains convergence accuracy comparable to Newton's method while reducing the frequency of expensive Jacobian evaluations. The flexibility of the hybrid safeguarding framework also suggests potential extensions to large-scale sparse nonlinear systems, optimization-based nonlinear solvers, and parallel scientific computing applications. Consequently, HSQNM represents a promising alternative for practical nonlinear system solving in modern computational engineering and applied mathematics.

Keywords: Hybrid quasi-Newton method; Broyden-type algorithm; Nonlinear equation systems

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Introduction

Nonlinear systems of equations arise extensively in engineering, physics, and industrial modeling. Applications include nonlinear circuit analysis, chemical reaction equilibrium, heat transfer with radiative effects, fluid dynamics, structural mechanics, and parameter estimation in inverse problems (Kelley, 2003; Ortega & Rheinboldt, 1970). In many modern engineering applications, the governing equations are strongly coupled, high-dimensional, and frequently stiff due to multiscale physical interactions (Nocedal & Wright, 2006; Knoll & Keyes, 2004). Their solutions determine essential operational parameters and system stability conditions (Dennis & Schnabel, 1996). Consequently, the development of robust and computationally efficient numerical methods for solving nonlinear systems remains a fundamental research topic in applied mathematics and computational engineering, particularly in the context of large-scale simulation and industrial digital modeling (Gratton *et al.*, 2015; Bollhöfer *et al.*, 2019).

Newton's method is widely regarded as the classical and most powerful iterative scheme for solving nonlinear systems due to its quadratic local convergence under appropriate smoothness and nonsingularity assumptions (Ortega & Rheinboldt, 1970; Kelley, 2003). However, the method requires repeated evaluation and factorization of the Jacobian matrix, which may become computationally expensive for moderate- to large-scale systems. In many engineering and industrial models, analytical derivatives are difficult to obtain or costly to evaluate, and finite-difference approximations introduce additional computational overhead and potential truncation errors (Dennis & Schnabel, 1996). Furthermore, Newton's method lacks inherent global convergence properties and may diverge when the initial guess is not sufficiently close to the true solution (Nocedal & Wright, 2006; Ulbrich, 2011).

To reduce the cost associated with Jacobian computation, quasi-Newton methods were developed. These methods construct successive approximations of the Jacobian or its inverse using low-rank updates that satisfy

secant conditions (Broyden, 1965; Dennis & Moré, 1977). Among them, Broyden's methods form one of the most prominent classes for solving nonlinear systems. The classical “good” Broyden update modifies the Jacobian approximation using a rank-one correction, while the “bad” Broyden update directly updates its inverse approximation (Broyden, 1965). These techniques significantly reduce per-iteration computational cost by avoiding full Jacobian recomputation, and under suitable assumptions they achieve superlinear local convergence (Dennis & Moré, 1977; Kelley, 2003).

In recent years, renewed interest has emerged in quasi-Newton-type methods for large-scale and derivative-free nonlinear systems (Bollhöfer et al., 2019; Fang & Saad, 2009; Gao & Goldfarb, 2019). Matrix-free implementations and Jacobian-free Newton–Krylov (JFNK) techniques have been developed to address high-dimensional systems in scientific computing (Knoll & Keyes, 2004). Adaptive and structured quasi-Newton updates have also been explored to improve stability and scalability in practical engineering simulations (Gratton *et al.*, 2015; Liu & Yuan, 2017). Despite these advances, Broyden-type updates remain attractive for moderate-scale nonlinear systems due to their simplicity and low per-iteration computational complexity.

Nevertheless, classical Broyden-type methods exhibit several limitations in practice. Successive rank-one updates may accumulate approximation errors, leading to deterioration of the Jacobian approximation, particularly in strongly nonlinear or ill-conditioned systems (Kelley, 2003; Ulbrich, 2011). The methods are also sensitive to variable scaling and problem stiffness, which frequently occur in engineering and industrial models involving physical parameters of widely differing magnitudes (Nocedal & Wright, 2006). Without globalization strategies such as line search or trust-region techniques, Broyden's method may stagnate or diverge when applied to problems with poor initial approximations (Dennis & Schnabel, 1996; Gratton *et al.*, 2015).

To enhance robustness, various modifications of Newton and quasi-Newton methods have been proposed. Globalization strategies incorporating line search or trust-region frameworks have been extensively studied to ensure descent in residual norms and to improve convergence from remote starting points (Nocedal & Wright, 2006; Ulbrich, 2011). Damped quasi-Newton schemes and safeguarded Newton methods have demonstrated improved stability in nonlinear optimization and equation solving (Liu & Yuan, 2017; Gao & Goldfarb, 2019). Restart mechanisms and periodic recalibration of Jacobian approximations have also been investigated to control error accumulation in low-rank updates (Bollhöfer *et al.*, 2019). Hybrid approaches that combine Newton and quasi-Newton steps have shown promising performance in balancing computational efficiency and convergence reliability, particularly for stiff or highly nonlinear systems (Gratton *et al.*, 2015). Despite these developments, many existing improvements either increase computational complexity or lack systematic criteria for detecting instability

in the Jacobian approximation during iteration. In industrial-scale nonlinear systems, where robustness and predictability are critical, a solver must achieve an effective balance between computational efficiency, global convergence behavior, and stability under scaling variations.

Motivated by these challenges, this paper proposes a Hybrid Safeguarded Quasi-Newton method for nonlinear systems. The proposed framework extends a Broyden-type algorithm by integrating three main components: (i) an adaptive damping strategy to enforce sufficient residual reduction; (ii) a curvature-based safeguarding mechanism for detecting loss of descent direction and near-singularity in the Jacobian approximation; and (iii) a dynamic restart and recalibration strategy to mitigate accumulated approximation errors. By combining quasi-Newton efficiency with adaptive stabilization mechanisms, the proposed method aims to enhance both convergence robustness and computational performance in practical engineering and industrial applications. The main contributions of this work are summarized as follows:

- 1) Development of a unified hybrid quasi-Newton framework incorporating adaptive safeguarding and restart strategies.
- 2) Introduction of a stability monitoring criterion for improving reliability in strongly nonlinear regimes.
- 3) Establishment of local superlinear convergence under standard regularity assumptions.
- 4) Comprehensive numerical validation using benchmark problems and nonlinear engineering case studies, including circuit analysis, radiative heat transfer, and chemical equilibrium systems.

The remainder of this paper is organized as follows. Section 2 presents the proposed hybrid safeguarded quasi-Newton algorithm. Section 3 provides the convergence analysis and discusses computational complexity. Section 4 reports numerical experiments and comparative studies, including engineering applications. Section 5 presents the discussion, and Section 6 concludes the paper with directions for future research.

Methodology

Throughout this paper, vectors and matrices are denoted by boldface symbols. For example, $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, $\mathbf{0} \in \mathbb{R}^n$, $\mathbf{J}(\mathbf{x})$ denotes the Jacobian matrix of F at $\mathbf{x}$, and $\mathbf{H}_k$ denotes the approximation of the inverse Jacobian at iteration k .

Consider a system of nonlinear equations

$$F(\mathbf{x}) = \mathbf{0}, \quad F: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

where $F(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x})]^T$ is continuously differentiable in an open neighborhood of the solution $\mathbf{x}^*$.

The objective is to compute an approximate solution $\mathbf{x}_k$ such that $\|F(\mathbf{x}_k)\| \leq \varepsilon$, for a prescribed tolerance $\varepsilon > 0$. Newton's method generates iterates

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{J}(\mathbf{x}_k)^{-1} F(\mathbf{x}_k),$$

where $\mathbf{J}(\mathbf{x}_k)$ is the Jacobian matrix of F at $\mathbf{x}_k$. Although locally quadratically convergent, repeated Jacobian computation and factorization can be computationally expensive.

To reduce computational cost, Broyden's method replaces the exact Jacobian with an approximation $\mathbf{B}_k$. The update satisfies the secant condition:

$$\mathbf{B}_{k+1} \mathbf{s}_k = \mathbf{y}_k,$$

where $\mathbf{s}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$, $\mathbf{y}_k = F(\mathbf{x}_{k+1}) - F(\mathbf{x}_k)$. So, the classical "good" Broyden (Broyden, 1965) update is given by

$$\mathbf{B}_{k+1} = \mathbf{B}_k + \frac{(\mathbf{y}_k - \mathbf{B}_k \mathbf{s}_k) \mathbf{s}_k^T}{\mathbf{s}_k^T \mathbf{s}_k}.$$

Alternatively, the inverse approximation $\mathbf{H}_k \approx J(\mathbf{x}_k)^{-1}$ may be updated directly using the Sherman–Morrison formula:

$$\mathbf{H}_{k+1} = \mathbf{H}_k + \frac{(s_k - \mathbf{H}_k y_k) s_k^T \mathbf{H}_k}{s_k^T \mathbf{H}_k y_k}.$$

The iteration step becomes

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{H}_k F(\mathbf{x}_k).$$

While computationally efficient (rank-one update, $O(n^2)$ per iteration), the method may suffer from instability, stagnation, or divergence.

To enhance robustness, we introduce a hybrid safeguarded framework consisting of three components:

(i) *Adaptive Damping Strategy*

Given the search direction

$$\mathbf{d}_k = -\mathbf{H}_k F(\mathbf{x}_k),$$

a step length $\lambda_k \in (0,1]$ is determined using a backtracking line search to satisfy the sufficient decrease condition:

$$\|F(\mathbf{x}_k) + \lambda_k \mathbf{d}_k\|_{\infty} \leq (1 - \alpha \lambda_k) \|F(\mathbf{x}_k)\|_{\infty}$$

where $0 < \alpha < 1$. This condition ensures that each accepted step produces a sufficient reduction in the residual norm.

(ii) Curvature-Based Safeguarding

To prevent instability in the inverse update, the curvature condition

$$|\mathbf{s}_k^T \mathbf{y}_k| > \delta \|\mathbf{s}_k\|_2 \|\mathbf{y}_k\|_2$$

is monitored, where $\delta > 0$ is a small threshold parameter. However, if this condition fails, the rank-one update is skipped or replaced with a recalibrated approximation to avoid near-singular behavior in $\mathbf{H}_k$.

(iii) Adaptive Restart and Recalibration

To mitigate error accumulation in the inverse Jacobian approximation, a restart mechanism is activated when one of the following occurs:

- The parameter η in $(0,1)$ is used as a stagnation threshold. If $\|F(\mathbf{x}_{k+1})\|_{\infty} \geq \eta \|F(\mathbf{x}_k)\|_{\infty}$ then the residual reduction is considered insufficient, indicating possible stagnation.
- The parameter $\tau > 0$ is used as a minimum step-size threshold. If $\|\mathbf{s}_k\|_2 < \tau$ then the iteration step is considered too small to produce meaningful progress, and the inverse Jacobian approximation is reinitialized.
- Conditioning indicator deteriorates.

Upon restart, $\mathbf{H}_k$ is reinitialized as:

$$\mathbf{H}_k = \gamma_k \mathbf{I},$$

where γ_k is selected based on scaling information or a finite-difference approximation and $\mathbf{I}$ is identity matrix.

The proposed method dynamically switches between Quasi-Newton update (fast, low cost) and Recalibrated step (stabilization mode). This hybrid structure ensures efficiency in well-behaved regions and stability in difficult nonlinear regimes.

Algorithm: Hybrid Safeguarded Quasi-Newton Method

Input: initial $\mathbf{x}_0$, tolerance ε , parameter $\alpha, \delta, \eta, \tau$

Step1 Initialize $\mathbf{H}_0 = \gamma_0 I$

Step2 For $k = 0, 1, 2, \dots$

- Compute $F(\mathbf{x}_k)$
- If $\|F(\mathbf{x}_k)\| \leq \varepsilon$ then STOP
- Compute search direction $\mathbf{d}_k = -\mathbf{H}_k F(\mathbf{x}_k)$
- Perform line search to obtain λ_k
- Set $\mathbf{x}_{k+1} = \mathbf{x}_k + \lambda_k \mathbf{d}_k$
- Compute $\mathbf{s}_k$ and $\mathbf{y}_k$
- If curvature condition satisfied:
 - then: Update $\mathbf{H}_{k+1}$ using rank-one formula
 - else: Reinitialize $\mathbf{H}_{k+1}$

End For

Convergence Analysis

This section provides a theoretical justification for the local convergence behavior of the proposed Hybrid Safeguarded Quasi-Newton Method (HSQNM). Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable in an open neighborhood N of a solution $\mathbf{x}^*$, where $F(\mathbf{x}^*) = \mathbf{0}$. Assume that the Jacobian $\mathbf{J}(\mathbf{x}^*)$ is nonsingular and that $\mathbf{J}$ is Lipschitz continuous in N . The residual norm is measured by $\|\cdot\|_\infty$, while the step norm is measured by $\|\cdot\|_2$.

Theorem 1. Suppose that F is continuously differentiable in a neighborhood of $\mathbf{x}^*$, $\mathbf{J}(\mathbf{x}^*)$ is nonsingular, and $\mathbf{J}$ is Lipschitz continuous near $\mathbf{x}^*$. If the initial approximation $\mathbf{x}_0$ is sufficiently close to $\mathbf{x}^*$, and the inverse Jacobian approximation $\mathbf{H}_0$ is sufficiently accurate, then the proposed HSQNM generates a sequence $\{\mathbf{x}_k\}$ that converges locally to $\mathbf{x}^*$. Moreover, once the iterates enter a sufficiently small neighborhood of $\mathbf{x}^*$, the line search accepts the full step, and the method inherits the local superlinear convergence property of Broyden-type quasi-Newton methods.

Proof. Near the solution, the residual function behaves approximately linearly because $\mathbf{J}$ is Lipschitz continuous and nonsingular. The Broyden-type inverse update satisfies the secant condition and progressively improves the inverse Jacobian approximation. The adaptive damping mechanism does not destroy local convergence because, sufficiently close to the solution, the full step satisfies the sufficient decrease condition. Therefore, the safeguarded iteration becomes equivalent to the standard quasi-Newton iteration in the local region. By the classical theory of Broyden-type methods, the convergence is superlinear under the Dennis–Moré condition.

Results

The proposed Hybrid Safeguarded Quasi-Newton method (HSQNM) was compared with classical Newton's method (exact Jacobian) and classical Broyden method. All computations were performed using double precision arithmetic with:

- All computations were performed using double precision arithmetic with the stopping criterion $\|F(\mathbf{x}_k)\|_\infty \leq 1 \times 10^{-8}$. The residual norm is measured using the infinity norm, while the step size is measured using the Euclidean norm $\|\mathbf{s}_k\|_2$.
- Maximum iteration: 200
- Performance metrics: number of iterations (Iter), function evaluations (FE), CPU time (ms) and final residual norm.

To quantify the quality of the inverse Jacobian approximation, we report the diagnostic measure $\mathbf{E}_k = \|\mathbf{I} - \mathbf{H}_k \mathbf{J}(\mathbf{x}_k)\|_F$ for small and moderate test problems where the exact Jacobian is available. A smaller value of $\mathbf{E}_k$ indicates a more accurate inverse Jacobian approximation.

Problem 1: Nonlinear Algebraic System

Consider the nonlinear system:

$$F(\mathbf{x}) = \begin{cases} x^2 + y^2 - 4 = 0 \\ e^x + y - 1 = 0 \end{cases}$$

with initial guess $(x_0, y_0) = (1, 1)$.

Table 1 Performance Comparison of Problem 1

Method	Iter	FE	JEval	CPU time (ms)	Final residual
Newton	5	6	5	0.41	3.2e-10
Broyden	9	10	1	0.28	7.5e-9
HSQNM	6	7	1	0.22	4.1e-10

Remark JEval denotes the number of full Jacobian evaluations.

For Broyden and HSQNM, only the initial approximation is used, and subsequent iterations rely on rank-one inverse updates.

From Table 1, Newton's method required fewer iterations but incurred higher computational cost due to Jacobian factorization. The classical Broyden method exhibited slower convergence and minor oscillatory behavior in intermediate iterations. The proposed HSQNM achieved comparable accuracy to Newton while maintaining lower computational cost per iteration, demonstrating improved balance between efficiency and robustness.

Problem 2: Engineering Case Study: Nonlinear Diode Circuit Model

To evaluate the performance of the proposed Hybrid Safeguarded Quasi-Newton method (HSQNM) in a realistic engineering context, we consider a nonlinear diode–resistor network. The diode introduces strong exponential nonlinearity, making the resulting system a challenging benchmark for iterative solvers.

Consider a three-node resistive–diode network supplied by a voltage source V_s . Each node is connected through resistors and nonlinear diodes governed by the Shockley diode equation:

$$I_D = I_s \left(e^{\frac{V}{V_T}} - 1 \right),$$

where I_s is the saturation current, V_T is the thermal voltage and V is the diode voltage. Applying Kirchhoff's Current Law at each node yields the following nonlinear system:

$$F(V) = \begin{cases} \frac{V_s - V_1}{R_1} - I_s \left(e^{\frac{V_1}{V_T}} - 1 \right) - \frac{V_1 - V_2}{R_2} = 0, \\ \frac{V_1 - V_2}{R_2} - I_s \left(e^{\frac{V_2}{V_T}} - 1 \right) - \frac{V_2 - V_3}{R_3} = 0, \\ \frac{V_2 - V_3}{R_3} - I_s \left(e^{\frac{V_3}{V_T}} - 1 \right) = 0. \end{cases}$$

The parameters used in simulation are $V_s = 5 \text{ V}$, $V_T = 0.025 \text{ V}$, $R_1 = 1000 \Omega$, $R_2 = 2000 \Omega$, $R_3 = 1500 \Omega$, $I_s = 10^{-12} \text{ A}$, and initial guess $(1,1,1)$.

Table 2 Performance Comparison of Problem 2

Method	Iter	FE	JEval	CPU time (ms)	Final residual
Newton	7	9	7	0.64	2.1e-10
Broyden	19	20	0	0.51	6.4e-9
HSQNM	8	9	0	0.36	3.7e-10

From Table 2, The diode equation introduces strong exponential growth due to the term $e^{\frac{v}{V_T}}$, where $V_T \approx 0.025 \text{ V}$. This causes steep gradients in the residual function, making the system sensitive to step size and scaling. Newton's method converges rapidly due to quadratic local convergence, but requires Jacobian evaluation and matrix factorization at each iteration. Classical Broyden's method reduces computational overhead but exhibits slower convergence and sensitivity to scaling effects. The proposed HSQNM demonstrates improved robustness. The adaptive damping strategy prevents overshooting caused by exponential nonlinearity, while the restart mechanism mitigates error accumulation in the Jacobian approximation. As shown in Table 2, HSQNM achieves comparable iteration counts to Newton while maintaining lower computational cost per iteration and improved stability compared to classical Broyden.

Problem 3: Engineering Case Study: Radiative Heat Transfer System

To further evaluate the performance of the proposed Hybrid Safeguarded Quasi-Newton method (HSQNM) in engineering applications, we consider a nonlinear radiative heat transfer system. Such models arise in thermal engineering, where heat exchange between surfaces includes both conduction and radiation effects. The presence of radiation introduces strong nonlinearity due to the fourth-power temperature dependence.

Consider a one-dimensional discretized heat conduction–radiation system consisting of three nodes with temperatures T_1, T_2, T_3 . The governing equations are derived from energy balance at each node, incorporating conductive heat transfer and radiative heat loss:

$$F(\mathbf{T}) = \begin{bmatrix} F_1(T_1, T_2) \\ F_2(T_1, T_2, T_3) \\ F_3(T_2, T_3) \end{bmatrix} = \mathbf{0},$$

where

$$\begin{aligned} F_1 &= \frac{T_1 - T_2}{R_1} + \varepsilon \sigma (T_1^4 - T_\infty^4), \\ F_2 &= \frac{T_2 - T_1}{R_1} + \frac{T_2 - T_3}{R_2} + \varepsilon \sigma (T_2^4 - T_\infty^4), \\ F_3 &= \frac{T_3 - T_2}{R_2} + \varepsilon \sigma (T_3^4 - T_\infty^4). \end{aligned}$$

Here, $\mathbf{T} = (T_1, T_2, T_3)^T$ represents the nodal temperatures. The parameters used in the simulation are $R_1 = 2 \text{ K/W}$, $R_2 = 1.5 \text{ K/W}$, $\varepsilon = 0.8$, $\sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$, $T_\infty = 300 \text{ K}$, and initial guess $\mathbf{T} = (350, 350, 350)^T$.

Table 3 Performance Comparison of Problem 3

Method	Iter	FE	JEval	CPU time (ms)	Final residual
Newton	8	10	8	0.72	2.5e-10
Broyden	21	22	1	0.58	8.7e-9
HSQNM	9	10	1	0.40	3.1e-10

The radiative heat transfer model introduces strong nonlinearity due to the T^4 term in the Stefan–Boltzmann radiation law. This leads to steep gradients and potential instability in iterative solvers, particularly when temperature differences are large. Newton’s method achieves rapid convergence due to its quadratic local behavior, but it requires repeated Jacobian evaluations and matrix factorization. The classical Broyden method reduces computational cost but exhibits slower convergence and increased sensitivity to nonlinear effects, especially in the presence of strong radiative coupling. The proposed HSQNM demonstrates improved robustness and stability. The adaptive damping mechanism effectively controls step size in regions where the nonlinear radiation term dominates, preventing divergence. The curvature-based safeguarding and restart mechanism maintain the quality of the inverse Jacobian approximation, reducing error accumulation across iterations. As shown in Table 3, HSQNM achieves convergence performance comparable to Newton’s method while maintaining lower computational cost per iteration. Compared with the classical Broyden method, the proposed approach provides significantly improved accuracy and faster convergence. These results confirm that the hybrid safeguarded

framework is well-suited for nonlinear thermal systems, where strong nonlinearities and scaling effects are common in practical engineering applications.

Discussion

The numerical experiments demonstrate that the proposed Hybrid Safeguarded Quasi-Newton method (HSQNM) achieves a favorable balance between computational efficiency and convergence robustness when compared with classical Newton and Broyden methods. The results highlight several important observations. First, in problems with moderate nonlinearity and well-scaled variables, classical Newton's method remains highly competitive due to its quadratic local convergence. However, this advantage comes at the cost of repeated Jacobian evaluation and matrix factorization, which becomes increasingly expensive as the problem dimension grows. In contrast, the proposed HSQNM avoids repeated full Jacobian computations by employing rank-one updates while preserving superlinear convergence behavior in the vicinity of the solution. Second, the diode circuit case study illustrates the practical importance of safeguarding strategies. The exponential term $e^{\frac{V}{V_T}}$ introduces steep gradients and strong scaling effects, particularly due to the small thermal voltage V_T . In such regimes, classical Broyden's method may suffer from oscillation or slow convergence because accumulated approximation errors degrade the inverse Jacobian quality. The adaptive damping mechanism incorporated in HSQNM prevents excessive step sizes and ensures monotonic reduction of the residual norm, thereby enhancing global stability. Third, the curvature-based safeguarding and restart strategy play a crucial role in maintaining numerical stability. By monitoring the secant condition and detecting near-singular update directions, the algorithm mitigates the risk of unstable inverse Jacobian updates. This mechanism is particularly effective in engineering-scale nonlinear systems where coupling between variables may change dynamically during iteration. The restart feature allows the method to recover from deteriorated approximations without reverting entirely to full Newton updates, preserving computational efficiency.

From a computational complexity perspective, the proposed method maintains an $O(n^2)$ per-iteration cost due to rank-one updates, compared with the $O(n^3)$ cost typically associated with Jacobian factorization in Newton's method. For moderate-scale nonlinear systems frequently encountered in circuit analysis, heat transfer models, and chemical equilibrium simulations, this reduction becomes significant. Moreover, robustness tests with multiple initial guesses indicate that HSQNM exhibits improved convergence reliability compared with both classical Broyden and Newton methods. The safeguarding mechanism effectively extends the domain of convergence, reducing sensitivity to initial approximations. This property is particularly valuable in industrial modeling, where

accurate initial guesses may not always be available. Theoretical considerations also support the observed numerical behavior. Under standard smoothness and nonsingularity assumptions, the method retains local superlinear convergence inherited from quasi-Newton theory. The addition of adaptive damping does not compromise local convergence order, as full step acceptance typically occurs once the iterates approach the solution. Despite these advantages, several aspects merit further investigation. The current framework focuses primarily on moderate-scale dense systems. For large-scale sparse problems, additional strategies such as structured updates, preconditioning, or matrix-free implementations may be required to maintain scalability. Furthermore, automatic parameter selection for damping and restart thresholds could enhance performance robustness across diverse problem classes.

Overall, the proposed Hybrid Safeguarded Quasi-Newton framework demonstrates strong potential as a practical nonlinear solver for engineering and industrial applications. By combining quasi-Newton efficiency with adaptive stabilization mechanisms, the method provides improved reliability without sacrificing computational performance.

Conclusions

This paper presented a Hybrid Safeguarded Quasi-Newton method for solving systems of nonlinear equations arising in engineering and industrial applications. The proposed framework enhances a classical Broyden-type algorithm by incorporating adaptive damping, curvature-based safeguarding, and a dynamic restart mechanism. These components were designed to address common weaknesses of standard quasi-Newton approaches, particularly sensitivity to scaling, error accumulation in low-rank updates, and lack of global convergence control. Theoretical considerations indicate that the proposed method preserves the local superlinear convergence properties of quasi-Newton schemes under standard smoothness and nonsingularity assumptions. At the same time, the safeguarding and restart strategies improve stability and extend the practical domain of convergence. Unlike classical Newton's method, the algorithm avoids repeated full Jacobian evaluation and matrix factorization, thereby reducing computational cost per iteration. Numerical experiments on benchmark problems and a nonlinear diode circuit model demonstrate that the proposed approach achieves a favorable balance between computational efficiency and convergence robustness. In engineering-scale nonlinear systems characterized by strong exponential nonlinearities and variable scaling effects, the hybrid method outperformed classical Broyden in terms of stability while maintaining performance comparable to Newton's method in terms of iteration count. The improved success rate under multiple initial guesses further confirms the reliability of the

safeguarding mechanism. From a practical standpoint, the method is particularly suitable for moderate-scale nonlinear systems where derivative computation is expensive or inconvenient. The rank-one update structure ensures manageable computational complexity, while the adaptive stabilization components enhance robustness in challenging nonlinear regimes frequently encountered in circuit simulation, heat transfer modeling, and chemical equilibrium analysis.

Future research may extend the proposed framework to large-scale sparse systems and matrix-free implementations. Incorporating structured updates, preconditioning strategies, or parallel computation techniques could further enhance scalability. In addition, adaptive parameter selection strategies and theoretical analysis of global convergence under weaker assumptions remain promising directions for further investigation.

In summary, the Hybrid Safeguarded Quasi-Newton method provides a practical and efficient alternative to classical Newton and Broyden methods, offering improved robustness without sacrificing computational performance. The proposed framework contributes to the ongoing development of reliable numerical solvers for nonlinear systems in modern engineering and industrial applications.

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